How Large Is The Inflation Risk Premium?

A Monetary Model of the Term Structure

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This Draft: 11 May 2001

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Abstract: The answer is: "60 basis points, on average over the last 40 years". However, the term structure of the risk premium is sharply upward sloping and it shows substantial time variation over the business cycle. The empirical result is obtained by estimating a general equilibrium production economy in which (i) the monetary policy is responsive to both nominal and real shocks, (ii) the stochastic process for inflation is endogenous, (iii) taxes are paid on nominal income and nominal capital gains. The fiscal system is imperfectly indexed to inflation shocks so that in equilibrium there may exist a positive inflation risk risk premium. (iv) In addition to nominal shocks, the stochastic investment opportunity set is affected by multiple real factors.

We obtain closed-form solutions for the equilibrium term structure of the nominal interest rates, of index linked bonds and of the risk premium on the inflation rate. The estimation is based all US Treasury bonds traded between 1960 to 2000. Moreover, we compare the extent of time variation of risk premia on nominal (monetary) shocks with respect to real (technological) shocks. The results shed some light on whether the rejection of the expectation hypothesis is due to the time variation of the risk premium on nominal shocks or on the time variation of the risk premium on the real factors, or both.

We are very grateful to Lars P. Hansen for several useful comments and suggestions on an earlier version of the paper, to an anonymous referee and to Geert Bekaert, David Chapman, John Cochrane, George Constantinides, Ian Cooper, Silverio Foresi, Rick Green, Stefano Risa, Nizar Touzi, Raman Uppal, seminar participants at Carnegie Mellon, The University of Chicago, Duke University, Insead, London Business School, University of Michigan, MIT Sloan School, Nortwestern University, UCLA, University of Rochester, Wharton, University of Southern California, the Annual Economic Conference at Abbazia di Mirasole, Milan, the Annual Conference in Quantitative Finance, Sidney, the 2000 Bachelier Conference, Paris, and the WFA 2000 Conference. Special thanks to the WFA Committee who awarded this paper the *Best Paper Award in Investment*.

1 Introduction

This paper analyzes the term structure of nominal yields when both expected and unexpected innovations in the price level can introduce distortions on the real capital accumulation. The paper is organized in two parts. In the first part, we describe a monetary economy in which taxes are raised on nominal profits. In this economy, the lack of a perfect indexation mechanism is responsible for deviations from the Fisher hypothesis. We derive closed form solutions for both the nominal and real term structure of interest rates and we characterize the risk premia. In the second part, we estimate the model and discuss the time varying properties of the risk premium on the inflation rate.

In the theory of asset pricing and of the term structure of interest rates, several important contributions have improved our understanding of the relation between asset prices and the underlying economic variables. Such contributions include the production economy of Cox, Ingersoll and Ross (1985), the consumption-based model of Breeden (1979), and term structure models such as Vasicek (1977) and Constantinides (1992). However, in most of these studies the monetary side of the economy plays little or no role and it is often common to assume an exogenous process for the inflation rate and to conjecture that the Fisher relation holds[1]. This contrasts with several empirical macroeconomic studies that find sharp deviations from the neutrality assumption. Some of the findings include: (a) The inflation rate is negatively related to the real interest rate, in terms both of realized changes and expected values. This effect is sometimes called the Tobin-Mundell effect due to the early studies of Mundell (1963) and Tobin (1965) who relate this effect to the role, in a partial equilibrium setting, of the expected inflation on the optimal portfolio composition between money and nominal bonds; (b) Real returns on nominal bonds decline when inflation increases (Fama (1981) and Fama and Gibbons (1982)); (c) Real returns on the stock market are negatively correlated with the inflation; (d) In the medium and long term, the real gross domestic product is negatively affected by an increase in inflation (Fama and Gibbons(1982), Boudoukh (1993), Harvey(1988)).

For this reason, we feel it is important to analyze in more detail some of the structural links between inflation and asset prices in a setting in which the previous empirical regularities are not ruled-out apriori. The link we focus on is generated by the role of the fiscal system. Intuitively, given a real pre-tax return on capital, if taxes and fiscal incentives are calculated on nominal historical values, then the inflation rate affects the after-tax real return on capital which, in turn, affects ex-ante decisions regarding the optimal allocation of (real) resources[2]. In most countries, the nominal nature of the fiscal system is quite pervasive:

a. Depreciation is usually computed on the nominal "historical" cost of the assets, rather than on the "current" nominal replacement value. Thus, an increase in expected inflation reduces the real after-tax return on capital and this reduces the optimal level

[1] An important exception is Bakshi and Chen (1996), who derive closed-form solutions for the term structure of interest rates in a monetary economy.

[2] Feldstein, Green and Sheshinki (1978), Feldstein (1980), Fisher and Modigliani (1978).

of investment in long-lived capital. In turn, this affects the equilibrium term structure of interest rates.

b. Capital gains are usually taxed in terms of their nominal value. Thus, an increase in inflation reduces the after-tax return on equity, given the pre-tax real return. This affects the optimal capital structure favoring forms of debt financing.

c. Interest payments on debt is usually deductible on their nominal values. Thus, an increase in inflation reduces the after-tax cost of debt-financing.

When the fiscal system is based on nominal and historic values, inflation becomes an important risk factor and plays a non-neutral role in the efficient allocation of resource. If this is the case, what are the implications on the term structure of interest rates?

In order to address this issue, in this paper we consider a monetary economy in which the fiscal system is only partially indexed to inflation shocks. The inflation affects the optimal allocation of resources and it gives rise to an endogenous risk premium on the inflation rate. We focus on two main issues:

(a) What is the size of the risk premium and how is it different across different maturities? Is it statistically and economically significant?

(b) Is the inflation risk premium time-varying? Well documented empirical evidence shows the existence of an general time-varying risk premium implicit in bond prices[3]. For example, Engle, Lilien and Robins (1987), using a Garch-in-mean specification, find evidence of a relation between the conditional variance of excess bond returns and expected excess returns[4]. Moreover, Bekaert, Hodrick and Marshall (1997), and several other empirical studies, reject the expectation hypothesis for the US term structure and interpret the rejection as a sign of a time-varying risk premium. In this paper we want to go a step further and ask, "Is the expectation hypothesis rejected because of time variation in the risk premia on real (technological) shocks or because of time variation in the risk premium on the inflation rate?"

In order to characterize a time varying inflation risk premium we specify a general equilibrium economy in which the inflation rate is endogenous and its equilibrium process has stochastic volatility and it can affect the real investment opportunity set. Since we want to focus on the inflation risk premium, we generalize the CIR[5] (1985 b) model to consider a monetary economy in which it is optimal, in equilibrium, to have positive monetary holdings. The equilibrium price process is affected by the monetary policy which sets the nominal

[3]Shiller, Campbell and Schoenhaltz (1983), Mankiw and Summers (1984), Shiller (1990) discuss the evidence against the expectation hypothesis of interest rates.

[4]Boudoukh (1993) estimates a vector autoregressive model with stochastic volatility and discusses the implications of inflation-output correlation on the time series properties of nominal bond prices.

[5]Duffie and Kan (1996) discuss the conditions necessary to obtain closed-form solutions to the yield curve. Closed form solutions cannot be obtained unless both the drift and the local volatility of the underlying factors are linear in the state variables. Examples of these processes include Ornstein-Uhlenbeck diffusions (Vasicek, 1977), square-root diffusions (Cox, Ingersoll and Ross 1985b) and translated square-root processes (Pearson and Sun, 1994).

money supply in response to deviations from long-terms objectives, expressed in terms of both inflation and economic growth. Thus, the real and nominal side of the economy are allowed to affect each other both from the demand and supply side. In this setting, we derive closed-form solutions for the term structure of nominal interest rates, of index linked bonds and of the risk premium on the inflation rate.

How does this paper relate to the previous literature? Bakshi and Chen (1996) study a monetary economy in which positive monetary holdings are supported in equilibrium and derive closed-form solutions for the nominal term structure of interest rates. The main differences in this model are: (a) the monetary policy is endogenous, so that the nominal money supply is allowed to change in response to deviations from monetary and real targets; (b) we introduce taxes in the model and we allow for an imperfect indexation mechanism to nominal shocks. The equilibrium process of the general price level is affected both by supply and demand factors, so that (c) the investment opportunity set is affected by inflation innovations so that there exists a (time-varying) risk premium on the inflation rate. The advantage of their approach is that they can model a non-constant velocity of money. Evans (1998) uses the U.K. market for index-linked bonds to obtain the real term structure and the risk premium. In our paper, we show how to identify and estimate these components even when a long history of index-linked bonds is not available[6]. We achieve this goal by solving a structural general equilibrium monetary model. Pennacchi (1991) studies a generalized Vasicék economy with constant volatility in which the time series for the expected inflation is obtained from survey data as opposed to being estimated. The main advantage of our model is that we solve for a monetary economy in which the risk premia are potentially time-varying[7]. Other important related papers include Longstaff and Schwartz (1992), the two factor Cox, Ingersoll and Ross (1985b), Constantinides (1992), Duffie and Kan (1996), Litterman and Scheinkman (1988), Franchot, Janci and Lacoste (1993). However, in all these papers there is no explicit account for the risk premium on the inflation rate. Danthine, Donaldson and Smith (1987) consider a stochastic production economy which is a generalization of the production economy by Sidrauski (1967). The main goal of their paper is to test whether Sidrauski's result on the superneutrality of money holds in an equilibrium with uncertainty. They consider a representative agent economy and employ a money-in-utility approach to ensure that money is held in the equilibrium. Danthine, Donaldson and Smith do not attempt to solve the model in closed-form and show the existence of the equilibrium under very general restrictions on the production technology and utility function. They show that money is non-neutral, however in their setting the extent of the deviation is small. The crucial difference of our production apparatus over that in Danthine, Donaldson and Smith is the introduction of taxes that are responsible for an equilibrium inflation risk premium. Moreover, the monetary authority in our economy sets the money supply endogenously, depending on a set of real and nominal targets. Danthine, Donaldson and Smith assume an

[6]The U.S. Treasury issued index-linked bonds for the first time in May 1996.

[7]Pennacchi (1991) estimates a homoskedastic VAR of instantaneous real interest rates and expected inflation based on a generalization of Vasicek model. The generality of his approach is limited by the homoskedasticity assumption which implies time invariant risk premia.

exogenous money supply process.

In the empirical section of the paper, we estimate an overidentified system of equations with cross-equation restrictions imposed on a panel data of nominal yields of nominal bonds, the inflation rate process and monetary holdings. The observable variables are assumed to be measured with error. We find that shocks to the expected price level have non-trivial effects on the level, the slope and the curvature of the inflation risk premium through their distortionary impact on the investment opportunities. A conditional test rejects the Fisher hypothesis, thus suggesting that standard empirical estimates of real term structure models with nominal bonds would be biased by the non neutrality of the price level.

The paper is organized as follows: in section 2, we set up the model and derive the equilibrium nominal term structure, the forward rate, the instantaneous interest rate and the conditional term premium. Section 3 describes empirical testable restrictions and the econometric method. Section 4 describes the dataset. Section 5 discusses a test of specification for latent factor term structure models. Section 6 presents the empirical results regarding the risk premia on the inflation rate. Section 7 discusses the market for index-linked bonds and the estimation results based on a datset that includes TIPS. Section 8 suggests an interpretation of the rejection of the expectation hypothesis. Section 9 concludes.

2 The Structure of the Economy

The economy we have in mind is one in which a single good is produced by a representative agent who can decide either to consume it or reinvest it in a constant return to scale production technology. Real monetary holdings are assumed to provide a transaction service as they reduce the total amount of resources needed to achieve a given level of net consumption. In this economy, money is held because of its positive marginal productivity in the "shopping technology".

Assumption 1. *The preferences $\mathcal{U}(t, X_t)$ of the representative agent are time separable and logarithmic in real net consumption holdings X:*

$$\mathcal{U}(X) = \int_0^\infty e^{-\rho t} \ln X_t \, dt \qquad [A1]$$

Assumption 2. *Real monetary holdings M_t^d provide a transaction service as they reduce the total amount of gross resources C_t needed to obtain a given level of net consumption C_t. We model this feature by including money in the utility function*

$$X_t = C_t \left(M_t^d\right)^\gamma \text{ with } 0 \leq \gamma \leq 1 \qquad [A2]$$

The service provided by monetary holdings has decreasing returns to scale, so that $\frac{\partial^2 X_t}{\partial M^2} < 0$. *When* $\gamma = 0$*, monetary holdings do not generate any transaction service.*

The gross rate of return on capital depends on multiple technological shocks. Part of the total output is absorbed by the public sector which levies taxes on nominal profits at a rate t. The remaining net capital is optimally allocated to consumption, real monetary holdings or reinvested. We do not model public expenditure. This can introduce additional distortions which may have important pricing implications. In this paper, however, we model the economy as if the public expenditure does not affect the distribution of wealth[8] and we focus instead on the pricing implication of the tax system.

Assumption 3. *The real after-tax capital accumulation process depends on a k-dimensional vector of technological shocks* $\mathbf{Y}_t$*. The cost structure of the production technology consists of (a) a depreciation (maintenance) cost and (b) a variable cost proportional to the output. The depreciation rate is assumed to be equal to* λ_m*. The relative importance of the two forms of costs depend on the specific production structure and technology, and they are pinned down by* λ_m and λ_s*. The real after tax return on capital can be allocated to consumption, real monetary holdings* m_t^d*, or it can be reinvested,* dK_t*:*

$$C_t dt + m_t^d dt + dK_t = \underbrace{K_t \mathbf{1}' d\mathbf{Y}_t}_{\text{total output}} - \underbrace{\lambda_m K_t dt}_{\text{depreciation}} - \underbrace{\lambda_s K_t \mathbf{1}' d\mathbf{Y}_t}_{\text{variable production cost}} - Taxes \qquad [A3]$$

The evolution of the total accumulated monetary stock is described by: $M_t^d = M_{t-dt}^d + m_t^d dt$

The assumption of variable costs is made for sake of generality and is not crucial for our model. All the following results can still be obtained in the case of $\lambda_s = 0$. The entrepreneur has access to a real storage technology and we study the extent of the welfare costs of inflation and the size of the inflation risk premium due to the nominal nature of the fiscal system. If the storage technology were nominal, the distortionary effect of inflation would be even larger. The fiscal system is described as follows:

Assumption 4. *The fiscal authorities impose taxes both on (a) operating income and (b) capital gains. All costs of the entrepreneur are tax-deductable. However, as in the US tax code, we assume that the tax shield is calculated with respect to the historical nominal value of the capital expenditure. Thus, the real value, as of time* $t + h$*, of the income tax liability is equal to*

$$\lim_{h \to 0} \tau_{pr} \underbrace{\left(K_t \mathbf{1}'(\mathbf{Y}_{t+h} - \mathbf{Y}_t) - \frac{p_t}{p_{t+h}} \lambda_m K_t h - \frac{p_t}{p_{t+h}} \lambda_s K_t \mathbf{1}'(\mathbf{Y}_{t+h} - \mathbf{Y}_t) \right)}_{\text{taxable income with expenses deducted at historical cost.}} \qquad [A4]$$

[8] An example would be the case in which the public expenditure is distributed in lump sums.

The capital gain tax is equal to

$$\underbrace{\lim_{h\to 0} \tau_{cg}\left(\frac{p_{t+h}-p_t}{p_{t+h}}\right)K_t}_{\text{capital gains tax}}$$

The value of depreciation allowances is based on the nominal "*historical*" cost of the assets rather than on their "*current*" nominal replacement value. Thus, an increase of the price level decreases the real value of depreciation so that the real value of taxable profits rises. The effect is larger for firms using the longest-lived capital, so that, through this channel, an increase in the expected inflation rate induces a decrease of the rate of investment and a shift to shorter-lived capital. The taxation of capital gains is an important source of real distortions caused by inflation. Given a constant real pre-tax return on investment, an increase of the inflation rate decreases the after-tax real return on the investment. Thus, under a nominal fiscal system, the equilibrium real marginal rate of transformation is affected by the inflation rate process thus having asset pricing implications.

Clearly, the assumption of a linear tax liability has some important limitations: (1) it does not capture the fact that negative tax liabilities are usually carried forward to future periods, when the company is not in a tax paying position. In this case, the value of a negative tax liability should be specified as the discounted value of the previous expression. (2) It implicitly rules out the existence of a tax timing option, as discussed in Green and Holifield (2001). However, the inclusion of these non-linear features would make the model untractable. The linear specification of the tax structure is necessary to have a constant return to scale production technology, which is needed in order to obtain closed-form solutions.

If we substitute these two tax terms in the capital accumulation process, given an equilibrium price $\frac{dp_t}{p_t}$, we obtain the capital accumulation process. It can be noticed that the inflation affects the after tax return on investment in several ways. (a) Given a positive capital gain tax τ_{cg}, the higher the inflation rate, the higher the real value of the tax liabilities because of the nominal capital appreciation. Thus, the higher the opportunity cost of capital investment. (b) The higher the inflation, the lower the real value of the costs that are deducted for tax reasons and the lower the real value of the capital that is depreciated. These effects make entrepreneurs using capital intensive technology averse to inflation shocks. Since τ_{cg} is about 30% in the US and even higher in other countries, inflation has a first order effect on the real accumulation of capital and therefore on equilibrium asset prices.

The greater the extent of indexation of the fiscal system, the smaller the relative size of the inflation tax. Thus, several countries have recently tried to limit the nominal nature of their fiscal system by introducing different forms of indexation[9]. When the fiscal system is

[9] In Chile and Israel, capital depreciation is computed with respect to the real book value of the assets.

perfectly indexed to the general price level, taxes are a function of *real* operating income, so that the capital accumulation process simplifies to:

$$\frac{dK_t}{K_t} = \left[dY_t \left(1 - \tau_{pr}\right) \left(1 - \lambda_s\right) - \lambda_m \left(1 - \tau_{pr}\right) dt - \frac{C_t}{K_t} dt - \frac{m_t^d}{K_t} dt \right]$$

The issue of the non-neutrality of nominal shocks has been debated in macroeconomics for a long time. However, empirical models of the term structure usually assume that the Fisher hypothesis holds[10]. Cox, Ingersoll and Ross (1985b) have 2 models with an exogenous process for inflation that satisfy, by construction, the property of Fisher-neutrality[11]. The innovation of this paper with respect to previous models is to relax the independence assumption in a flexible but technically tractable way.

Assumption 5. *The real before-tax marginal productivity of capital depends on a vector of production factors, here described by the following stochastic differential equations:*

$$\begin{aligned} dy_t^i &= z_t^i \mu_y^i dt + \sigma_y^i \sqrt{z_t^i} dW_t^{y_i} && 1 \le i \le k \\ dz_t^i &= \left(\xi^i z_t^i + \zeta^i\right) dt + \sigma_z^i \sqrt{z_t^i} dW_t^{z_i} && 1 \le i \le k \qquad [A5] \\ E(dW_t^{y^i}, dW_t^{z^i}) &= \rho_{y^i, z^i} dt && 1 \le i \le k \end{aligned}$$

The drift μ_y^i in equation $[A5]$ makes the process of the stock of capital non-stationary (consistently with the empirical evidence), while the state variables z_t^i are still characterized by unconditionally stationary distributions. In the spirit of Longstaff and Schwartz (1992), this set-up is flexible enough to capture differential effects of the state variables z_t^i, for $i \le k$, on the marginal productivity of capital. If $\mu_y^i \neq 0$ and $\sigma_y^i = 0$, the state variable z_t^i affects only the instantaneous return on capital and not its local variance. If $\mu_y^j = 0$ and $\sigma_y^j > 0$, the state variable z_t^j affects the local variance without changing the local mean of the return on capital. Thus, σ_y^i is related to the uncertainty (unexpected innovations) of the marginal productivity of capital, while σ_z^i characterizes the volatility of expected innovations in the marginal productivity of capital. In this sense, given estimates of the parameters μ_y^i and σ_y^i we can relate the effects of innovations in the pricing factors to the dynamics of the productivity of capital. For the sake of generality, we allow the two Brownian motions $dW_t^{y_i}$ and $dW_t^{z_i}$ to be potentially correlated. Cox, Ingersoll and Ross (1985 a) assume that the economic agent can form a portfolio of different production processes. Since it can be shown that in this economy the optimal portfolio composition is time invariant, with no

[10] An important exception is Pennacchi (1991). He generalizes the Vasicek model to distinguish between real and nominal interest rates. He uses survey data to identify inflationary expectations. Since in his Vasicek-type model the factors are Ornstein-Uhlenbeck processes, his model is exposed to the usual criticism that nominal and real interest rates can become negative and that the volatility is constant.

[11] Other papers in which the inflation rate is considered as an independent and neutral process are, among others, Gibbons and Ramaswamy (1993), Pearson and Sun (1993), Chen and Scott (1993).

loss of generality equation $[A5]$ can be interpreted as the evolution equation of the marginal productivity of the optimal linear combination of the original production activities.

From the dynamics of capital, it is possible to see that realized returns on physical investment dK_t/K_t are affected by the stochastic evolution of technological shocks $d\mathbf{Y}_t$, by the equilibrium price process p_t^* and by their covariance.

The nominal side affects the real allocation of resources, thus equilibrium asset prices and risk premia, both because of expected changes in the price level and also because of the volatility of inflation. The first effect has already been described. Equation $[A3]$ shows that the higher the volatility of inflation, the higher the volatility of the real capital accumulation process. A higher level of uncertainty on the future productivity, due to higher volatility, decreases the optimal investment in real capital.

The policy function of the monetary authorities is defined as follows:

Assumption 6. *The monetary authority sets the money supply M_t^s on the basis of three nominal and real economic targets: (i) A long-term target for the growth of nominal money supply equal to $-\theta/k$; (ii) an inflation target equal to $\overline{\pi}$, and (iii) an economic growth rate equal to $\bar{\kappa}$. Short term deviations from the optimal long-run level of money growth are allowed to have level-dependent time-varying volatility. These properties are summarized as follows*

$$\begin{aligned} \frac{dM_t^s}{M_t^s} &= v_t dt + q_1\left(\frac{dK_t^*}{K_t^*} - \bar{\kappa}dt\right) + q_2\left(\frac{dp_t^*}{p_t^*} - \overline{\pi}dt\right) + \sqrt{\sigma_{0M}^2 + \sigma_{1M}^2 v_t}\, dW_t^M \qquad [A6] \\ dv_t &= (kv_t + \theta)dt + \sqrt{\sigma_{0v}^2 + \sigma_{1v}^2 v_t}\, dW_t^v \end{aligned}$$

Since the monetary aggregate is controlled by the monetary authorities only imperfectly by use of intermediate instruments, we model the evolution of the monetary aggregate as a stochastic process.

When $q_1 = q_2 = 0$, the monetary policy is exogenous and v_t is equal to the expected nominal money growth. When $q_1 = 0$, real productivity shocks feedbacks to the nominal side of the economy, not just because of the market clearing condition for monetary holdings, but also because the central monetary authority reacts to deviations from the long-term economic growth target $\bar{\kappa}$. The intensity of the adjustments to the long-term real and nominal targets are given by the parameters q_1 and q_2. When σ_{1v}^2 and σ_{1M}^2 are different from zero, the conditional volatilities of the monetary shocks are not constant. In this case, the risk premium on the nominal factor can be time varying, which, in principle, may account for the rejection of the expectation hypothesis so frequently found in the empirical literature.

We want to solve for the equilibrium consumption level, real monetary holdings and price process $\{C_t^*, M_t^{*d}, p_t^*\}$ such that the representative agent maximizes its expected utility, given the price process and market clearing condition $p_t^* M_t^{*d} = M_t^s$.

First, let us solve for the optimal policy functions $\{C_t^*, M_t^{*d}\}$ of the representative agent, given p_t^*. Then, let us solve for the equilibrium stochastic process of the general price level that (i) satisfies the market clearing condition for monetary holdings and (ii) is consistent with the budget constraint of the representative agent.

A necessary condition for the policy functions of the representative agent to be optimal is the existence of a value function $J(K_t, \mathbf{z}_t, v_t)$ with respect to which the policy functions are solutions of the following Hamilton-Bellman-Jacobi programming problem[12]:

$$-\frac{\partial}{\partial t}J(K_t, \mathbf{z}_t, v_t) = \max_{\{c_t, M_t^d\}} \{\mathcal{U}(X) + \mathcal{A}J(K_t, \mathbf{z}_t, v_t)\}$$

subject to $X_t = C_t(M_t^d)^\gamma$, the capital accumulation process $[A3]$, the process for the productivity shocks $[A5]$ and the monetary policy $[A6]$.

The following proposition summarizes the equilibrium solutions of the optimal consumption level, real monetary holdings and price process.

Proposition 1 (The Equilibrium) *Given the previous description of the economy, the following results follow:*

a. The value function of the representative agent is

$$\begin{aligned} J(t, K_t, \mathbf{z}_t, v_t) &= e^{-\rho t} J(K_t, \mathbf{z}_t, v_t) \\ &= \frac{1}{\rho} e^{-\rho t} \left[P + Q \ln(\rho K_t) + \sum_{i=1}^{n} R_{z^i} z_t^i + R_v v_t \right] \end{aligned}$$

b. The agents optimally allocate a constant fraction of wealth to consumption, $C_t^* = \frac{\rho}{Q} K_t$ and to real monetary holdings $M_t^{*d} = \gamma \frac{\rho}{Q} K_t = \gamma C_t^*$, with $Q = 1 + \gamma$.

c. The equilibrium process for the endogenous general price level that clears the money market is:

$$\frac{dp_t^*}{p_t^*} = \mu_{p^*}(z_t^i, v_t)dt + \sigma_{p^*}(z_t^i, v_t)dW_t^{p^*} \tag{1}$$

where the stochastic drift and variance are linear functions in the state variables $\{z_t^i, v_t\}$:

$$\begin{aligned} \mu_{p^*}(z_t^i, v_t) &= \mu_0^{p^*}(\underline{q}, \underline{\tau}, \overline{k}, \overline{\pi}) + \mu_{z^i}^{p^*}(\underline{q}, \underline{\tau}, \overline{k}, \overline{\pi}) z_t^i + \mu_v^{p^*}(\underline{q}, \underline{\tau}, \overline{k}, \overline{\pi}) v_t \\ \sigma_{p^*}(z_t^i, v_t) dW_t^{p^*} &= \frac{(q_1 - 1)(1 - q_2)\sigma_{y^i}}{(1 - q_2) - \tau_{cg}(1 - q_1)} \sqrt{z_t^i} dW_t^{y^i} - \frac{2\tau_{cg}(q_1 - 1) + (1 - q_2)}{(1 - q_2) - \tau_{cg}(1 - q_1)} \sqrt{\sigma_{0M}^2 + \sigma_{1M}^2 v_t} dW_t^M \end{aligned}$$

The full specification of the drift parameters $\mu_0^{p^*}$, $\mu_{z^i}^{p^*}$ and $\mu_v^{p^*}$ is given in the Appendix.

[12] We will use $\mathcal{A}\phi$ to denote the differential operator applied to the function $\phi(X)$,

$$\mathcal{A}\phi(X) = \nabla_x \phi(X) \mu_x(X) + \frac{1}{2} Tr \{\nabla_{xx} \phi(X) \cdot \Sigma_x(X)\}$$

with X being a multidimesional Ito process $dX = \mu_x(X)dt + \Sigma_x(X)dW$.

d. The equilibrium stochastic process of the real stock of capital is:

$$\frac{dK_t^*}{K_t^*} = \mu_{K^*}(z_t^i, v_t)dt + \sigma_{K^*}(z_t^i, v_t)dW_t^{K^*}$$

where the stochastic drift and variance are linear functions in the state variables $\{z_t^i, v_t\}$:

$$\begin{aligned} \mu_{K^*}(z_t^i, v_t) &= \mu_0^{K^*}(\underline{q}, \underline{\tau}, \overline{k}, \overline{\pi}) + \mu_{z^i}^{K^*}(\underline{q}, \underline{\tau}, \overline{k}, \overline{\pi})z_t^i + \mu_v^{K^*}(\underline{q}, \underline{\tau}, \overline{k}, \overline{\pi})v_t \\ \sigma_{K^*}(z_t^i, v_t)dW_t^{K^*} &= \left[\frac{(1-q_2)}{(1-q_2)+\sigma_{y^i}(1-q_1)}\right]\left[-\tau_{cg}\sqrt{z_t^i}dW_t^{y^i} + \frac{\sigma_{y^i}}{(1-q_2)}\sqrt{\sigma_{0M}^2 + \sigma_{1M}^2 v_t}dW_t^M\right] \end{aligned}$$

The full specification of the drift parameters $\mu_0^{K^*}$, $\mu_{z^i}^{K^*}$ *and* $\mu_v^{K^*}$ *is given in the Appendix.*

The proof of Proposition 1 is given in the Appendix. The main point of the derivation is that, in equilibrium, both the process of the general price level p_t^* and of the stock of capital K_t^* are jointly determined. This is due to the fact that the inflation rate is required to clear the money market in any state of the world and that if τ_{pr} or τ_{cg} are different from zero, nominal shocks have non-neutral implications on the real allocation of resources due to the lack of perfect indexation of the fiscal system. Moreover, when q_1 or q_2 are also different from zero, the monetary authority pursues an active economic policy that depends on the extent of the deviations from the real and nominal long-term economic targets.

Let us turn now to describe the equilibrium value of the term structure of interest rates. From the first order conditions of the representative agent, the price of a nominal zero coupon bond B_t^τ, with time to maturity τ, is equal to the conditional expected value of the intertemporal marginal rate of (consumption) substitution multiplied by the real payoff at maturity of the bond:

$$B_t^\tau = E_t\left[e^{-\rho\tau}\frac{\exp\left(-\ln X_{t+\tau}^*\right)\frac{1}{p_{t+\tau}^*}}{\exp\left(-\ln X_t^*\right)\frac{1}{p_t^*}}\right] \tag{2}$$

Substituting the optimal consumption schedule in (2) and defining, for convenience, $\kappa_t^* = (\gamma+1)\ln K_t^* + \rho t$, the equilibrium term structure of interest rates is the solution of the following stochastic problem:

$$B_t^\tau = \frac{1}{\exp\left(-\kappa_t^*\right)\frac{1}{p_t^*}}E_t\left[\exp\left(-\kappa_{t+\tau}^*\right)\frac{1}{p_{t+\tau}^*}\right] \tag{3}$$

It should be noticed that the vector diffusion process of $\{\kappa_t^*, p_t^*\}$ is non stationary. However, in this case the Yamada-Watanabe Theorem guarantees the existence of a weak solution unique in probability law for the stochastic process $\exp(-\kappa_t^*)\frac{1}{p_t^*}$. Moreover, since the growth conditions for the unbounded function $\exp(-\kappa_t^*)\frac{1}{p_t^*}$ are satisfied[13], we can apply a generalized version of the Feynman-Kac Theorem and obtain closed-form solutions for (3) by solving the

[13] A formal discussion of these conditions can be found in the Appendix.

dual representation of (3) expressed in differential form. Given equilibrium levels of the stock of capital, the general price level, and the vector of state variables $(\kappa_t^*, p_t^*, v_t^*, \mathbf{z}_t^*)$, if $B(\kappa_t^*, p_t^*, v_t^*, \mathbf{z}_t^*; \tau)$ is solution to the stochastic problem (3), then it must also be solution of the following differential problem:

$$\begin{aligned} \frac{d}{dt} B(\kappa_t^*, p_t^*, v_t^*, \mathbf{z}_t^*; \tau) &= \mathcal{A} B(\kappa_t^*, p_t^*, v_t^*, \mathbf{z}_t^*; \tau) \\ s.t. \quad B(\kappa_t^*, p_t^*, v_t^*, \mathbf{z}_t^*; 0) &= 1, \quad \forall t \; and \; \tau \end{aligned}$$

The solution of the previous partial differential equation is summarized in the following Proposition.

Proposition 2 (The Nominal Term Structure) *The nominal price of a nominal zero coupon bond B_t^τ, with time to maturity τ, is a log-linear function of the real productivity and nominal shocks z_t^i, v_t. The closed-form solution is:*

$$B_t^t(\kappa_t, p_t, v_t, \mathbf{z}_t; 0) = A(\tau) \exp\left[-b_v(\tau) v_t - \sum_{i=1}^{n} b_{z^i}(\tau) z_t^i \right] \tag{4}$$

$$A(\tau) = \exp(A_0 \tau) a_v(\tau) c_v(\tau) \prod_{i=1}^{n} a_{z^i}(\tau)$$

Let us define $j = v,\ z^i$ then

$$\begin{aligned}
b_j(\tau) &= \tfrac{1}{2\Theta_2^j} \left[-\Theta_1^j + \sqrt{D^j} \tan\left(\arctan(\tfrac{\Theta_1^j}{\sqrt{D^j}}) - \tfrac{1}{2}\tau\sqrt{D^j} \right) \right], \mathit{where}\, D^j = -(\Theta_1^j)^2 + 4\Theta_0^j \Theta_2^j \\
a_j(\tau) &= 2^{\frac{A_j}{\Theta_2^j}} \exp(-\tfrac{A_j \tau \Theta_1^j}{2\Theta_2^j}) \left[\cos(\arctan(\tfrac{\Theta_1^j}{\sqrt{D^j}}) - \tfrac{1}{2}\tau\sqrt{D^j}) \right]^{\frac{A_j}{\Theta_2^j}} \left(\tfrac{\Theta_2^j \Theta_2^j}{D^j} \right)^{\frac{A_j}{2\Theta_2^j}} \\
c_v(\tau) &= \exp\left[\tfrac{1}{2\Theta_2^v} \left(B_v \Theta_2^v + B_v \tau \Theta_2^\tau - 2 B_v \tau \Theta_2^v \Theta_2^v - B_v \Theta_1^v \log(1 + \tfrac{\Theta_2^v}{D^v}) \right) \right] \\
&\quad \cdot \exp\left[\tfrac{1}{2\Theta_2^v} \left(-B_v \sqrt{D^v} \tan\left[\arctan(\tfrac{\Theta_1^v}{\sqrt{D^v}}) - \tfrac{1}{2}\tau\sqrt{D^v} \right] \right) \right] \cdot \left[\cos\left(\arctan(\tfrac{\Theta_1^v}{\sqrt{D^v}}) - \tfrac{1}{2}\tau\sqrt{D^v} \right) \right]^{-\frac{B_v \Theta_1^v}{(\Theta_2^v)^2}}
\end{aligned}$$

$\{\Theta_0^j, \Theta_1^j, \Theta_2^j, A_0, A_j, B_v\}$ *are functions of the structural parameters of the economy, the parameters controlling for the extent of non-neutrality of the fiscal system, $\underline{\tau}$ and the monetary policy parameters $\underline{q}$. The proof of the Proposition and the explicit functional forms of* $\{\Theta_0^j, \Theta_1^j, \Theta_2^j, A_0, A_j, B_v\}$, *in terms of the structural parameters, can be found in the Appendix.*

The previous equilibrium pricing equation shows some interesting features of the term structure of discount bond prices. (i) The yield curve, defined as $-\frac{1}{\tau} ln B_t^\tau$, is linear in the state-variables. This property is induced by the linearity of the local variance in the pricing factors and it is shared by models such as Cox, Ingersoll and Ross (1985b), Vasiček (1977) and others. (ii) The long-term monetary targets $\overline{\pi}$ and $\overline{\kappa}$ affect the intercept of the yield curve but not the slope with respect to the factors. (iii) The parameters $\underline{\tau}$, which describe the extent of non-neutrality, and the monetary policy parameters $\underline{q}$ affect both the intercept and the slope of the yield curve. (iv) The previous equilibrium pricing equation can accommodate a great variety of yield curves. Each single pricing factor can have a different effect on the local mean and local variance of the marginal productivity of capital. Thus, depending on the values of the structural parameters, a positive shock to z_t^i can vary the equilibrium risk premium of discount bonds by changing independently the degree of uncertainty of the level of production or its instantaneous rate of return. This property will turn out to be very important in order to fit the term structure of conditional volatilities. Depending on the level of z_t^i, the yield curve can be monotone increasing, monotone decreasing. In addition, it can have a hump, a through or both. This characteristics is shared by multi factors models of term structure such as Longstaff and Schwartz (1992) and Constantinides (1992) but not by simpler single state variable models such as Vasiček (1977), Dothan (1978) and CIR (1985).

2.1 The Market Risk Premium

Duarte (2000), Dai and Singleton (2001), Backus, Telmer and Wu (1999) show that when the market risk premium is assumed to be proportional to the local volatility of the factors, several reduced form affine models find difficult to fit the empirical properties of the conditional second moments of interest rates. For instance, some of these model cannote duplicate the magnitude of the empirical violations of the expectation hypothesis. They suggest that the structural specification of the economy should give rise to a reduced form pricing equation with a richer time-varying and state depedent market risk premium. Thus, it is important to notice that, in our structural model, the market risk premium is not directly proportional to the volatility of the factors driving the dynamics of the term structure. To see yhis, let m_t be the stochastic discount factor, $m_t = \beta^t U'(c_{t,}^* M_t^*)$, so that the discounted value of any tradable asset is a martingale, $p_t m_t = E_t(m_{t+1} p_{t+1})$. It is well known that the diffusion process of the stochastic discount factor must be of the form:

$$\frac{dm_t}{m_t} = -r_t dt - \mathbf{\Lambda}_t d\mathbf{W}_t$$

with $\Lambda_t dW_t$ being the price of risk. If we solve for $\Lambda_t dW_t$ in the structural model, we have:

$$\mathbf{\Lambda}_t d\mathbf{W}_t = \frac{1+\gamma}{(1-q_2)+B_q(1-q_1)} \begin{bmatrix} (1-q_2)B_z\sqrt{z_t^i}dW_t^{y^i} \\ \\ B_p\sqrt{\sigma_{0M}^2+\sigma_{1M}^2 v_t}dW_t^M \end{bmatrix}$$

Thus, if $\frac{\sigma_{0v}^2}{\sigma_{1v}^2} \neq \frac{\sigma_{0M}^2}{\sigma_{1M}^2}$ the price of risk is not proportional to the volatility of the nominal factor, which is equal to $\sqrt{\sigma_{0v}^2+\sigma_{1v}^2 v_t}dW_t^v$. The action comes from the distinction between uncertainty in the *expected component (drift)* of the marginal productivity of capital dY_t and of the money supply process dM_t^s and uncertainty in the *unexpected component (level)* of dY_t and of dM_t^s. This breaks the relation of proportionality between the volatility of the factors and the risk premium, which is typical of many traditional affine specifications. Dai and Singleton (2001) show different generalizations of reduced form affine models that generate more flexibility in the market price of risk which are consistent with the empirical evidence. Our structural specification of the economy give rise endogenusly to a state-dependent price of risk that captures some of the features of their reduced form generalizations. Thus, it may have better chances to be consistent with the properties of the second moments of interest rates.

2.2 The Risk Premium on the Inflation Rate and the Term Structure of Index-Linked Bonds

When the inflation rate affects the real capital accumulation process, the term structure of nominal bonds includes a risk premium on the inflation rate. The term structure of inflation risk premia is given by $Cov_t\left[\frac{e^{-\rho\tau}U'(X^*_{t+\tau})}{U'(X^*_t)};\frac{p^*_t}{p^*_{t+\tau}}\right]$ and the relation between the prices of nominal and index-linked bonds, IL^τ_t, is given by

$$B^\tau_t = IL^\tau_t \cdot E_t\left(\frac{p^*_t}{p^*_{t+\tau}}\right) + Cov_t\left[\frac{e^{-\rho\tau}U'(X^*_{t+\tau})}{U'(X^*_t)};\frac{p^*_t}{p^*_{t+\tau}}\right]$$

The closed-form solution of the term structures of index-linked bonds and of inflation risk premia are summarized in the following Proposition:

Proposition 3 (The Term Structure of Risk Premia)
(a) The term structure of index-linked bonds *and the expected value of the reciprocal of the equilibrium rate of inflation are both affine in the nominal (monetary) and real (productivity) factors v_t and z^i_t.*

$$\begin{aligned} IL^\tau_t &= A^{IL}(\tau)\exp\left\{-b^{IL}_v(\tau)v_t - \sum_{i=1}^n b^{IL}_{z^i}(\tau)z^i_t\right\} \\ E_t\left(\frac{p^*_t}{p^*_{t+\tau}}\right) &= A^p(\tau)\exp\left\{-b^p_v(\tau)v_t - \sum_{i=1}^n b^p_{z^i}(\tau)z^i_t\right\} \end{aligned}$$

The parameters $A^{IL}(\tau), b^{IL}_v(\tau), b^{IL}_{z^i}(\tau)$ and $A^p(\tau), b^p_v(\tau), b^p_{z^i}(\tau)$ of the closed-form solution are functions of the structural parameters of the economy, the parameters controlling for the extent of non-neutrality of the fiscal system $\underline{\tau}$ and the monetary policy parameters $\underline{q}$. The explicit functional forms are provided in the Appendix.

(b) From property (a), it follows that the term structure of inflation risk premia *is not affine. Its closed-form solution is given by:*

$$Cov_t\left[\frac{e^{-\rho\tau}U'(X^*_{t+\tau})}{U'(X^*_t)};\frac{p^*_t}{p^*_{t+\tau}}\right] = B^\tau_t - IL^\tau_t \cdot E_t\left(\frac{p^*_t}{p^*_{t+\tau}}\right)$$

The shape of the term structure of risk premia is clearly very important for capital budgeting purposes. The closed-form solution of the inflation risk premium enables us to estimate the entire term structure of inflation risk premia, whose shape is sensitive to the extent of indexation to nominal shocks and to the responsiveness of the monetary authority to deviations from the monetary targets. It enables us to study the empirical differences between the short and long-term inflation risk premia. We will explore in more depth these properties in Section 6 of the paper.

How does the instantaneous spot interest rate change in response to nominal shocks? Do the overnight nominal rate increases one-to-one with inflation? The short answer is no. In order to study in detail this relationship, let us use the equilibrium solution for the price of a nominal bond to compute the instantaneous interest rate of the monetary economy. From the definition of the instantaneous spot interest rate, the *nominal* instantaneous interest rate is:

$$\begin{aligned} r_t &= -\frac{d}{d\tau}B_t^\tau\bigg|_{\tau=0} = -\frac{d}{d\tau}\ln B_t^\tau\bigg|_{\tau=0} \\ &= \frac{-A'(\tau)}{A(\tau)}\bigg|_{\tau=0} + \sum_i b'_{z^i}(\tau)|_{\tau=0}\, z_t^i + b'_v(\tau)|_{\tau=0}\, v_t \end{aligned}$$

Since $A(\tau)$, $b_{z^i}(\tau)$ and $b_v(\tau)$ must be consistent with the terminal (par) value of the bond price, namely $B_t^0 = 1$, it is easy to verify that the following restrictions are satisfied by the solution for B_t^τ in Proposition 2:

$$A(0) = 1, \quad b_v(0) = 0, \quad b_{z^i}(0) = 0 \qquad \forall v_t, z_t^i$$

Moreover, since $\frac{A'(\tau)}{A(\tau)} = A_0 + A_v b_v(\tau) + \sum_{i=1}^n A_{z^i} b_{z^i}(\tau)$, it must be the case that $\frac{A'(\tau)}{A(\tau)}\Big|_{\tau=0} = A_0$. Since $b(\tau)$ satisfy the equation $-b'_v(\tau) = \Theta_0 + \Theta_1 b_v(\tau) + \Theta_2 b_v^2(\tau)$, then the following condition must be satisfied: $b'_v(0) = -\Theta_0$. Imposing these restrictions, we obtain:

$$r_t = -A_0 - \sum_i \Theta_0^{z^i} z_t^i - \Theta_0^v v_t \tag{5}$$

where the full specification of $A_0(\cdot)$, $\Theta_i^{z^i}(\cdot)$ and $\Theta_i^v(\cdot)$ for $i = 0, 1, 2$ as functions of the structural parameters, the monetary policy parameters q, $\overline{\pi}$, $\overline{\kappa}$ and the distortionary impact of the inflation, indexed by α, are given in the Appendix.

The inflation rate enters in a non-trivial way in the closed form solution of the nominal instantaneous interest rate. Since the inflation rate can distort the real capital accumulation process, nominal shocks may affect the nominal term structure both by changing the term structure of expected price levels and the real yield curve. Deviations from the Fisher neutrality due to the level of inflation rate and to changes in the volatility of inflation affect both the intercept and the slope of the spot rate schedule in different ways. The sensitivity of the spot interest rate with respect to v_t is affected by the extent of heteroskedasticity of the inflation rate, namely σ_{1v}^2. The intercept of the instantaneous interest rate is affected only by σ_{0v}^2, and not by σ_{1v}^2. Thus, derivative products that are more sensitive to changes in the slope of the term structure, such as Constant Maturity Swaps and Reverse Floaters, are particularly sensitive to the extent of heteroskedasticity of the monetary policy and of productivity shocks.

3 Econometric Methods

In this section, we use the restrictions obtained in Propositions 1 and 2 to estimate and test the overidentified representation of the economy. We estimate the structural parameters using a panel data of nominal bond yields and the price process and estimate the price of inflation risk.

The estimation of the structural parameters is based on the maximum likelihood approach proposed by Chen and Scott (1993). The procedure assumes that the covariance matrix of the measurement errors is not of full rank, so that one can use a subset of bonds to reveal the unobservable risk factors by inverting the pricing equation. In what follows, we present the three sets of moment conditions that are used in the maximum likelihood estimation method. These moment conditions refer to the yields, the inflation rate and the growth in monetary holdings, $[y_t^s, \ln \frac{p_{t+1}}{p_t}, \ln \frac{M_{t+1}^s}{M_t^s}]$. The closed form solution for the expected equilibrium values are given by the functional forms $[\mathcal{M}^y, \mathcal{M}^p, \mathcal{M}^M]$. In what follows, we describe the specification of these three moment conditions.

- The bond pricing equation can be written as:

$$y_t^\tau = \mathcal{M}^y(\underline{z}_t, v_t, \theta_o, \tau) + \eta_t^{y^\tau}, \qquad E\left[\eta_t^{y^\tau} | z_t^i, v_t\right] = 0 \tag{6}$$

where y_t is the observed *bond-yield process* with time to maturity τ, $\mathcal{M}(\theta_o)$ is the moment restriction with $\mathcal{M}^y : \mathcal{R}^k \times \mathcal{R}^p \rightarrow \mathcal{R}^{n_y}$, $\theta_o \in \Theta \subset \mathcal{R}^p$ is the vector of the structural parameters of the data generating process and η_t is the $(n \times 1)$ vector of measurement errors. From Proposition 2, it follows that:

$$\mathcal{M}^y(\underline{z}_t, v_t, \theta_o) = -\frac{1}{\tau} \ln A(\tau, \theta_0) + \frac{1}{\tau} b_v(\tau, \theta_0) v_t + \frac{1}{\tau} \sum_i b_{z^i}(\tau, \theta_0)\, z_t^i \tag{7}$$

- The endogenous stochastic process of the price level depends both on monetary and productivity shocks.

  Let us split the log increments of the price process into two orthogonal components, $\mathcal{M}^{p^*}$ and η^{p^*}, namely the expected and unexpected inflation rates, conditional on $\{z_t^i, v_t\}$.

$$\frac{1}{T-t} \ln \frac{p_T^*}{p_t^*} = \mathcal{M}^{p^*}(\underline{z}_t, v_t, \theta^p) + \eta_t^{p^*}, \qquad E\left[\eta_t^{p^*} | z_t^i, v_t\right] = 0 \tag{8}$$

  Thus, from Proposition 1 $(\mu^{p^*}, \sigma^{p^*})$ is a known function of the structural parameters. Using Ito's Lemma the instantaneous drift of the equilibrium log-price process is $\mu^{\ln p^*} \equiv \mu^{p^*} - \frac{1}{2}(\sigma^{p^*})^2$, which can be decomposed as the sum of three terms, $\mu^{\ln p^*} \equiv \mu_0^{\ln p^*} + \mu_{z^i}^{\ln p^*} z_t^i + \mu_v^{\ln p^*} v_t$. The solution of the conditional expected value at a one month frequency is given by

$$\mathcal{M}^{p^*}(\underline{z}_t, v_t, \theta^p) = \left[\mu_0^{\ln p^*} - \frac{\varsigma^i}{\xi^i}\mu_{z^i}^{\ln p^*} - \frac{\theta}{k}\mu_v^{\ln p^*}\right] + \mu_{z^i}^{\ln p^*}\frac{\left(e^{\xi^i(T-t)} - 1\right)}{\xi^i(T-t)}\left[z_t + \frac{\varsigma^i}{\xi^i}\right] + \quad (9)$$

$$\mu_v^{\ln p^*}\frac{\left(e^{k(T-t)} - 1\right)}{k(T-t)}\left[v_t + \frac{\theta}{k}\right] \quad (10)$$

which constitute our second moment restriction.

- The monetary authority follows an active monetary policy that is function of a nominal and a real economic target. Let us express the equilibrium money supply as a function of two terms: the expected and unexpected innovation, given $\{z_t^i, v_t\}$. The diffusion process for the money supply equation can be easily obtained by substituting the equilibrium values for the endogenous variables $\frac{dK^*}{k^*}$ and $\frac{dp^*}{p^*}$ in the policy function. Since the capital and price processes are affine (Proposition 1), the instantaneous drift $\mu^M$and volatility $\sigma^M$of the money supply process are affine in the nominal factors. Let us define $\mu^{\ln M^*} \equiv \mu^{M^*} - \frac{1}{2}(\sigma^{M^*})^2$, and let us decompose $\mu^{\ln M^*}$ in the sum of three terms, $\mu^{\ln M^*} \equiv \mu_0^{\ln M^*} + \mu_{z^i}^{\ln M^*} z_t^i + \mu_v^{\ln M^*} v_t$. Then, let us solve for the conditional expected value of monetary holdings. This constitutes our third empirical moment restriction.

$$\frac{1}{T-t}\ln\frac{M_T^s}{M_t^s} = \mathcal{M}^M(\underline{z}_t, v_t, \theta^M) + \eta_t^M, \qquad E\left[\eta_t^M | z_t^i, v_t\right] = 0 \quad (11)$$

with

$$\mathcal{M}^M(\underline{z}_t, v_t, \theta^M) = \left[\mu_0^{\ln M^*} - \frac{\varsigma^i}{\xi^i}\mu_{z^i}^{\ln M^*} - \frac{\theta}{k}\mu_v^{\ln M^*}\right] + \mu_{z^i}^{\ln M^*}\frac{\left(e^{\xi^i(T-t)} - 1\right)}{\xi^i(T-t)}\left[z_t + \frac{\varsigma^i}{\xi^i}\right] \quad (12)$$

$$+\mu_v^{\ln M^*}\frac{\left(e^{k(T-t)} - 1\right)}{k(T-t)}\left[v_t + \frac{\theta}{k}\right] \quad (13)$$

We follow the estimation methodology suggested by Chen and Scott (1993) and invert the pricing equations for a subset of bonds to reveal the unobservable risk factors $[\underline{z}_t, v_t]$. This procedure assumes that the covariance matrix of the measurement errors is less than full rank.

Given $[\underline{z}_t, v_t]$ expressed as a function of a subset of obesrvable variables and the structural parameters, we estimate the model by quasi maximum likelihood assuming that the measurement errors have zero mean and using the correct moments of $\left[y_t^\tau, \ln\frac{p_{t+\tau}^*}{p_t^*}, \ln\frac{M_{t+1}}{M_t}\right]$, which come from the general equilibrium solutions of the model.

The local volatility of v_t is $\sqrt{\sigma_{0v}^2 + \sigma_{1v}^2 v_t}$, so that when $\sigma_{1v} = 0$ the nominal factor is a mean-reverting Ornstein-Uhlenbeck process as in the Vasicek (1977) model; when $\sigma_{1v} = 0$ the nominal factor is a standard CIR factor. This adds an important element of flexibility that is needed in order to fit the moment conditions. This is discussed also in other papers, such as Backus, Telmer and Wu (1999) who say "it is difficult to reconcile [the dynamics of the term structure] when both state variables are required to be positive". They also add, "the properties of bond yields call for a model in which there is interaction between the factors". We achieve this by using a mixed Vasicek-CIR specification of the nominal factor and by having the monetary policy to be responsive to both nominal and real shocks.

The conditional second moment of the nominal factor v_t is somewhat different from traditional CIR factors. Since the drift is linear, the first conditional moment is independent of the shape of the local volatility, thus it is identical to the first conditional moment of a standard CIR factor, i.e. . $E_t(v_T) = v_t e^{k(T-t)} + \frac{\theta}{k}(e^{k(T-t)} - 1)$. The conditional variance is different. However, it can be easily computed by forward integration. Consider the canonical representation of the diffusion process of $d(v_t^2)$:

$$v_T^2 = v_t^2 + \int_t^T [2kv_u^2 + (2\theta + \sigma_{1v}^2)v_u + \sigma_{0v}^2]du + \int_t^T \sqrt{\sigma_{0v}^2 + \sigma_{1v}^2 v_u} dW_u$$

Taking the conditional expectation and using Fubini's theorem and the law of iterated expectations, we have

$$E_t\left(v_T^2\right) = v_t^2 + \int_t^T [2kE_t(v_u^2) + (2\theta + \sigma_{1v}^2)E_t(v_u) + \sigma_{0v}^2]du$$

Taking the partial derivatives with respect to T,

$$\frac{\partial}{\partial T} E_t\left(v_T^2\right) = 2kE_t(v_T^2) + (2\theta + \sigma_{1v}^2)E_t(v_T) + \sigma_{0v}^2$$

Since $E_t(v_T)$ is known, we can solve the previous ordinary differential equation with respect to $E_t(v_T^2)$, subject to $E_t(v_t^2) = v_t^2$. The solution is

$$\begin{aligned} E_t\left(v_T^2\right) &= v_t^2 e^{2k(T-t)} + \left[\frac{1}{k}(e^{2k(T-t)} - e^{k(T-t)})\right](2\theta + \sigma_{1v}^2)(v_t + \frac{\theta}{k}) + \\ &\quad + \left[\frac{1}{2k}(e^{2k(T-t)} - 1)\right]\left[\sigma_{0v}^2 - (2\theta + \sigma_{1v}^2)\frac{\theta}{k}\right] \end{aligned}$$

The conditional variance follows easily from $Var_t(v_T) = E_t(v_T^2) - E_t^2(v_T)$.

The transition density of the nominal factor is a shifted Chi-square distribution. Let us define $f_t = \sigma_{0v}^2 + \sigma_{1v}^2 v_t$, it is easy to see that the conditional distribution of f_t is a non-central Chi-square. Let us denote the cumulative distribution of the original process as $G(\overline{v}) = \Pr(v_T < \overline{v}|v_t)$. Thus, the distribution of the nominal factor can be obtained from the non-central χ^2 by taking a linear transformation of its argument, $\Pr\left(\frac{f_T - \sigma_{0v}^2}{\sigma_{1v}^2} < \overline{v}|f_t\right) = \chi^2\left(\sigma_{1v}^2\overline{v} + \sigma_{0v}^2\right)$.

4 The Dataset

The empirical results are based on 492 monthly observations from January 1960 to December 2000. The dataset consists of three main components: interest rate data, price levels data, and money supply data. Interest rate data from January 1960 to February 1991 are obtained from the McCulloch and Kwon dataset[14]. This database contains end-of-month zero-coupon yields and forward curves based on the McCulloch (1975) methodology from one month to 10 years. We extend this dataset using the daily GovPX dataset which provides end-of-day prices for all Treasury securities. The data is based on the transactions by the primary dealers through five of the six inter-dealer brokers for all active and off-the-run US Treasury. We keep the methodology for the construction of the zero-coupon yield curve as close as possible to that of McCulloch (1990) and Kwon (1992). We select the last business day of each month and we remove all callable bonds from consideration. At each day, the number of Treasury securities in the McCulloch dataset increases from slightly over 40 in the 1950s to over 200 in late 1980s. The average number of Treasury securities in each cross-section of our implied spot curves is 134 and it ranges from 100 to 200.

Inflation data is based on the Consumer Price Index (CPI) for all urban consumers which is available since January 1947. The money supply data used in this study is from the official H.6 release of the Federal Reserve Board of Governors. The data provided starts from January, 1959. We choose M2 as measure of the money stock as it most closely represents the notion of money in our model. It includes money market deposit accounts, which can be used for purchasing products and services, thus being consistent with our definition of money in the model. For our purposes, M3 is too wide as a measure of the money stock since it includes instruments that pay significant interest rates and they can not be classified as money in our framework.

Summary statistics for our sample are given in Table I, while Table II presents the correlation matrix. We find that the correlation between M2 growth and the yield on the 5 year zero coupon bond is 20%. Moreover, the monthly correlation between M2 growth and inflation is 16.8%. The relatively high correlation between money growth and both interest rates and inflation highlights the importance of considering explicitly the monetary side of the economy in order to explore the properties of the term structure of interest rates. Figure 2 plots the dynamics of the yield curve since January 1960. From the plot we can see periods in which the yield curve has double peaks, a situation that can not be explained by standard single factor models of the term structure.

Table I and II, *about here*

Figure 3 shows the behavior of the spread between the one year nominal interest rate and inflation. Gray boxes on the graph show the periods of US recessions compiled and reported by NBER. This spread, which can be considered as a rough proxy for real interest rates[15], declines during recessions. This is consistent with the fact that the return on capital

[14] See McCulloch(1990) and Kwon (1992).

[15] This proxy would be correct only if the risk premia were equal to zero.

falls during recessions, thus also the real interest rate. Expansion periods, on the contrary, enjoy high level of real interest rates. The only exception from this pattern is the 1981-1982 recession in which the real interest rate proxy remained at relatively high levels. A reason may be due to an increase, during the recession, of the risk premium. However, in order to test this hypothesis we need to estimate the time varying risk premium.

Figure 2 and 3, *about here*

5 Model Specification: Testing for the Number of Factors

Before estimating the model and discussing the results, we need to characterize the number of factors needed to describe the dynamics of the economy. In what follows, we discuss the methodology that we use to run the first set of test of model specification.

Several empirical papers have explored the performance of multiple factor models based on measures of goodness-of-fit. However, no formal statistical tests have been provided. The econometric difficulty of developing a test for the number of statistically relevant factors is that under the null hypothesis of interest (namely that the n^{th} factor is statistically irrelevant) the parameters that characterize the conditional density of the additional factor are not identified. The intuition is simple: they are present only under the alternative hypothesis. In other words, the asymptotic distribution of chi-square type tests is, under the null hypothesis, degenerate.

In what follows, we discuss a test of the number of latent factors for non-Gaussian continuous-time models of the term structure. Let

$$x_t = \mathcal{M}\left(\alpha_{1t}(\theta_1)'\theta_{1,0} + \alpha_{2t}(\theta_2)'\theta_{2,0}\right) + \varepsilon_t$$

where x_t is the endogenous variable and $\mathcal{M}$ is the vector of moment conditions. Let α_t be a vector of non Gaussian latent factors, $\theta = [\theta_1, \theta_{1,0}, \theta_2, \theta_{2,0}]$ a vector of parameters that characterizes the conditional densities of the state variables, with $\alpha_t|\alpha_{t-1} = [\alpha_{1,t}|\alpha_{1,t-1}; \alpha_{2,t}|\alpha_{2,t-1}]$. The question of interest is whether the latent factor α_{2t} is statistically significant, that is whether $H_0 : \theta_{2,0} = 0$.

If θ_2 were known, a standard Likelihoood Ratio or Wald test could be easily constructed. When θ_2 is unknown, either θ_2 is chosen in an arbitrary way, or if it is chosen in a data-dependent fashion the asymptotic distribution of the standard test statistics would be, under the null hypothesis, incorrect since θ_2 does not appear in the measurement equation when $\theta_{2,0} = 0$. Under the null hypothesis, the asymptotic distribution is degenerate. If we are interested in comparing the performance of the unrestricted model with n explanatory factors with the restricted model with $n-1$ factors, we must construct a test that does not require the a-priory knowledge of θ_2. Andrews and Ploberger (1994) and Hansen (1996) propose a test statistics that can be adapted to solve explicitly this issue.

The intuition is simple. Since a LM or Wald test statistic $W(\theta_2)$ can be constructed if the structural parameters θ_2^o of the unidentified latent variables were known, then Andrews and Ploberger (1994) and Hansen (1996) suggest to consider a weighted average of Wald test statistics, conditional on different values for θ_2^o:

$$\mathcal{G}(W) = \int W(\theta_2) dh(\theta_2)$$

In this case, the asymptotic distribution of the test statistics is well defined also under the null hypothesis and, although not known in closed form, it can be obtained to any degree of accuracy using Monte Carlo methods[16].

We use our estimated three factor model as a base case and test whether a two or three factor model is rejected versus the alternative of a three and four factor model, respectively. The two restricted parameters are μ_{y^i} and σ_{y^i}. When these are equal to zero, the parameters that describe the dynamics of the factors, namely ξ, ζ, and σ_z and ρ_{y_2,z_2} are not identified. This is responsible for the stochastic singularity of a standard Wald tests statistics when applied to a test of the number of factors in a term structure model.

The results of the two factors versus three factors model are reported in the upper panel of Table III[17]. We find that the null hypothesis of two factor model (one nominal factor and one real factor) is reject by the data. The p-value for the test statistic is less that one percent.

When we test the three versus four factors model, see lower panel of Table III, we find that the three factor specification (two real factors and one nominal factor) is not rejected by the data. The results are significant at the conventional 5 percent confidence level. In what follows, we will focus on this specification to report and discuss the empirical properties of the inflation risk premium.

Table III, *about here*

[16] In the interest of space we refer to B. Hansen (1996) for a detailed description of the Monte Carlo method used to compute the asymptotic distribution.

[17] We decide to report the results for only one of the two Andrews and Ploberger (1994)'s statistics, namely the "exponential" test statistics. Andrews and Ploberger (1994) find that the two statistics proposed by them are asymptotically equivalent. The small sample results are not qualitatively different and they are available upon request.

6 Empirical Results

6.1 Parameter Estimates

In this section, we consider a three factor model with two real and one nominal factor. The risk factors are modeled as latent state variables. The estimation method is based on the maximum likelihood approach by Chen and Scott (1993) which assumes that the covariance matrix of the observation errors is not of full rank. In this case, a subset of yields can be used to reveal the risk factors by inverting the pricing restrictions of the model. For this purpose, we use yields on the 3 months, 3 years and 10 years zero coupon bonds. The remaining maturities used for the estimation are the 1 and 6 months and the 1, 2, 5, 7 years.

The overall mean and median absolute errors of the term structure fitting are 16.9 and 12.6 basis points, respectively. This is extremely good, if one were to compare it to the 18 basis points[18] that Chen and Scott (1993) report for their three factor Cox, Ingersoll and Ross (1985) model. It is extremely good since we fit not only the term structure but also the inflation rate and monetary holdings. Moreover, we fit the term structure up to a maturity of 10 years, as opposed to 5 years as in Chen and Scott (1993). The mean absolute error for a maturity up to 5 years is 15.5 basis points.

Table IV, *about here*

Three main reasons account for the better empirical performance of this model. First, the capital evolution process is more general than in a standard CIR model since it includes a stochastic drift in the marginal productivity of capital. Second, we model the nominal side of the economy and the nominal state variables are shifted square root processes which have a different conditional distribution than the original CIR square root processes. Third, and perhaps most importantly, the tax implications of the nominal side of the economy have a crucial effect on the dynamics of the nominal term structure.

Estimates of the parameters of the model and their corresponding standard errors[19] are presented in Table V.

Table V, *about here*

The fiscal side of the economy is captured by τ_{pr} and τ_{cg}, the income and capital gain tax respectively. The estimated values of these parameters are 21.7% and 26.8% respectively and both of them are statistically significant at usual confidence levels. The estimated value of the capital gain tax is very close to the actual historical value of the tax rate; on the other hand, the income tax rate is substantially smaller than the statutory income tax rate. However, the estimated value of 21.7% is very close to the Effective Income Tax rate computed by the

[18] page 25 in Chen and Scott(1993).

[19] The asymptotic covariance matrix of the parameters is based on the outer-product of the Jacobian of the log-likelihood function.

Institute on Taxation and Economic Policy[20]. The ITEP estimated that during the 1981-1985 Reagan administration, the Effective Income Tax rate was 14.3%, substantially smaller than the 45% statutory tax rate. "The 1986 Tax Reform Act [...] closed several corporate loopholes and put almost all of the tax freeloaders back on the tax rolls. By 1988, a similar survey of large corporations found that the effective tax rate was back up to 26.5%." In the nineties, the Effective Tax rate ranged between 20.1% and 22.9%, with respect to a statutory rate of 35%, see Table VI. R. McIntyre and Coo Nguyen (2000) discuss in detail the major tax lowering items that are responsible for the difference between the statutory rate and the effective rate.

Table VI, *about here*

The long term capital gain tax is currently 20%. However, in the period $1987 - 1996$ and before 1980, the tax rate was 28%. The result is quite close to the implied capital gain tax rate estimated empirically.

The depreciation parameter λ_m is 11%, however the confidence interval is relatively high. The parameter that captures the variable cost component of investment, λ_s, is equal to 5.5% and it is significantly different from zero at standard confidence levels.

Estimates of θ and k can be used to calculate the expected long term growth of the money supply, which equals to the long run expected value of the nominal factor $-\theta/k$ or about 9 percent. Long run inflation target $\overline{\pi}$ of the monetary authority is found to be about 4.3 percent. The long-run real growth target $\bar{\kappa}$ is 3.4%.

6.2 The Term Structure of Volatility

In what follows we consider a set of tests of model specification based on the term structure of volatility as an additional set of moment restrictions. We use the closed-form solution for the yield volatility to construct a GMM test. The results shows that the model is able to fit the term structure of the *level* of volatility quite well, see Table VII. The additional cross-section restrictions imposed by the general equilibrium model are compatible with reasonable levels of the term structure of volatility. The empirical one month yield volatility is 2.594, compared to a fitted value of 2.446. A GMM test of the null hypothesis that the model-implied term structure of yield volatility is equal to its empirical counterpart is not rejected, with an overall p-value of 0.65. The overall declining pattern of the term structure is quite well captured by the model with one exception: the term structure of volatility induced by the model is monotone and it does not display the hump shape that characterizes the term structure for maturities shorter than six months.

Table VII, *about here*

[20] R. McIntyre and Coo Nguyen (2000).

We also test the ability of the model to reproduce the volatility of yield *changes*. Along this dimension, we cannot reject the null hypothesis in four out of nine cases. In absolute terms, at a five years horizon the model implied volatility of yield changes is 0.277 versus an empirical volatility of 0.389. From a statistical perspective, the performance of the model is significantly worse for intermediate maturities. This result is not surprising as it seems intrinsic to the class of affine models as discussed in Dai and Singleton (2001), Backus, Telmer and Wu (1999). Clearly, a specification of the local volatility that is linear in the factors, while necessary in order to retain tractability, is far too simple in order to capture the second moments of yield changes. From a risk management perspective, this should be a concern. It may be argued, however, that for the purpose of estimating the inflation risk premium, it is relatively more important to have an accurate estimate of the volatility of the *level* of yields.

With regards to the ability of the model of reproducing the conditional volatility of yields, we consider the methodology outlined in Chan, Karolyi, Longstaff and Sanders (1992), CKLS thereafter, and implemented also in Duarte (2000). We estimate the following restriction

$$\left(y_{t+\Delta t}^{n} - E_t\left[y_{t+\Delta t}^{n}\right]\right)^2 = \alpha + \beta E_t\left[\left(y_{t+\Delta t}^{n} - E_t\left[y_{t+\Delta t}^{n}\right]\right)^2\right] + \varepsilon_t$$

where the expectation is taken under the model objective probability measure, and test the null hypothesis that $H_0 : \alpha = 0$ and $H_0 : \beta = 1$. Moreover, one can estimate what proportion of the total variation in interest rate changes can be explained by the model.

The results of the test are mixed, see Table VIII. The null hypothesis that $H_0 : \alpha = 0$ is generally rejected, especially at a frequency of six months. However, the null hypothesis that $H_0 : \beta = 1$ are not rejected and the p-values range from 0.152 to 0.979 for the one year yield, depending on the conditioning horizon. Duarte (2000) run a similar test on a three factor CIR model and on his own model with a flexible specification of the price of risk. Based on the 1983-1998 sample period[21], he rejects the null hypothesis that $H_0 : \beta = 1$ and reports R^2 that range between 0.07 and 0.15 for the CIR model. We repeat the estimation on the same sample period. We cannot reject the null hypothesis that $H_0 : \alpha = 0$ in 25 out of 30 cases and $H_0 : \beta = 1$ in 20 out of 30 cases. The R^2 are similar to the one reported by Duarte[22]. The proportions of the time variability of the conditional volatility that we can explain is comparable to similar models of the term structure, so that our ability to match the inflation process does not seem to come at the expense of matching the time variability of the volatility of interest rates.

Table VIII, IX *about here*

[21] The important feature of this period is that it does not include the period of high interest rate volatility which is more difficult to predict.

[22] The existing difficulty to fit conditional volatilities of short term yields is documented in Piazzesi (2000). She suggests using jump-diffusions to generate more dynamics on the short side of the term structure. We feel that this would be beyond the scope of this paper.

6.3 Real Rates

As Figure 4 shows, during the 1960-2000 period the estimated short-term real interest rate ranges between -2% to 5%. The average short term real rate is 2%, while the average long-term real rate is 2.5%. Moreover, we find that the term structure of the volatility of real rates is sharply downward sloping, consistently with the small correlation coefficient between short and long maturity index-linked bonds. We find the short term real interest rates to drop during recessions. The long term real rates are, on the other hand, relatively stable and close to 2.5%, as implied by the sharply downward sloping term structure of volatility for real rates.

Figure 4, *about here*

Table X presents the correlation matrix of the estimated values of the nominal interest rate, the real interest rate, the risk premium and the expected inflation rate.

Table X, *about here*

One of the most significant results is the negative correlation between the real interest rate and the expected inflation, which is -48.9% at a 10 years horizon. This value compares to 60.9% for the correlation coefficient between the nominal interest rate and inflation. A similar result is found, in a related but different framework, by Pennacchi (1991) who suggests that the link between the nominal and the real economy should not be ignored by asset pricing models[23]. However, an important difference is that he finds a relatively larger volatility for the real rate, which is equal to 0.06[24] as opposed to 0.011 in our framework. A possible explanation is due to the fact that in Pennacchi (1991) the risk premia are assumed to be constant, so that all the volatility of the spread between nominal yields and expected inflation is interpreted as real rate volatility. In our framework, part of the time variation of this spread is due to the time variation in the risk premium.

We also find that the inflation risk premium is positively correlated with the level of inflation, with a correlation coefficient of 44% at a 10 year horizon and it is negatively correlated with the real interest rate.

On average, across the sample, the term structure of the general equilibrium expected values of inflation is found to be downward sloping. Such evidence of mean reversion is consistent with the specification of the monetary supply process that includes a nominal long term target.

[23] Another example of the non trivial link between real and nominal financial variables is given by Brown and Schaefer (1994). They compare British index-linked "real" bonds and nominal bonds find that the correlation coefficient of short-term and long-term British index-linked "real" bonds is between 0.5 and 0.6, while for nominal bonds is 0.9. Moreover, also the term structure of volatility is different as the conditional volatility sharply decreases as maturity increases in real bonds, while it decreases much less in the case of nominal bonds.

[24] see Pennacchi (1991), page 78.

6.4 The Expected Inflation Rate

Let us define $E_t(\pi_{t+1}|\, I_t)$ the model expected inflation. If the model is correctly specified, then the prediction errors should be orthogonal to any function of x_t, measurable with respect to I_t. If the model is not correctly specified, one may be able to improve on the model forecast using some function of the explanatory variable $\phi(x_t)$, i.e. $E_t(\pi_{t+1}|\, I_t) + \theta' \phi(x_t)$. Let us consider the inflation forecast error $u_{t+1} = \pi_{t+1} - E_t(\pi_{t+1}|\, I_t) - \theta' \phi(x_t)$, we study the null hypothesis $H_0 : \theta = 0$. The estimation and tests statistics are obtained using a standard GMM framework with the following moment conditions:

$$E[h(x_t, \theta)] = 0$$

with $h(x_t, \theta)$ being

$$h(x_t, \theta) = \begin{bmatrix} u_{t+1}(\theta) \\ u_{t+1}(\theta) \otimes [\phi(x_t)] \end{bmatrix}$$

It is well known that a consistent and efficient estimator of $\hat{\theta}$ is obtained as

$$\hat{\theta} = \arg\min_{\theta} [T \cdot h_T(x_t, \theta)]' W_T^{-1} [h_T(x_t, \theta)]$$

where $h_T(x_t, \theta) = \sum_{t=1}^{T} h_t(x_t, \theta)$ and the weighting matrix W_T is the Newey-West (1987) heteroskedastic and autocorrelation consistent estimator of the unconditional covariance matrix. A standard test of the null hypothesis $H_0 : \theta = 0$ can be constructed from the following statistics d_T

$$d_T = T \cdot \left[h_T(x_t, \theta = 0)' W_T^{-1} h_T(x_t, \theta = 0) - h_T\left(x_t, \hat{\theta}\right)' W_T^{-1} h_T\left(x_t, \hat{\theta}\right) \right]$$

which is distributed as a χ^2 under the null hypothesis.

We compare the performance of the model-implied expected inflation with the performance of two publicly available inflation forecasts, provided respectively by the Federal Reserve Bank of Philadelphia and the University of Michigan. The data provided by the Fed of Philadelphia is the expected price change for the following 4 quarters and it is available at a quarterly frequency. The value is calculated as the median value from the Survey of Professional Forecasters compiled by the Fed of Philadelphia. The data provided by the University of Michigan is compiled from the Survey of Consumers.[25].

Several popular macro models assume that the inflation rate follows a random walk. Such an assumption is motivated by several empirical studies. Thus, in what follows we also explore the difference in performance of the structural model with respect to the random walk hypothesis.

At a one month horizon, the null hypothesis that lagged values of inflation are orthogonal to the model prediction errors are rejected in the case both of the random walk specification

[25] Interestingly, we found that inflation forecasts from different agencies vary quite substantially. The difference between inflation forecasts from The University of Michigan and from the Philadelphia Bureau of Forecasting is well beyond the fitting errors of our model.

and of our structural model[26]. However, the extent of the deviation is much larger in the case of the random walk with a d_T statistics equal to 72.167 versus 15.538.

At a 12 months horizon, the null hypothesis that the survey-based prediction errors are orthogonal to lagged inflation is strongly rejected both in the case of survey-based forecasts of the Philadelphia Fed and of the survey-based forecasts of the University of Michigan, with p-values less than 1%. At a 12 months horizon, the structural model outperforms all the other three models in terms of orthogonality tests and the structural model is the only one to survive the orthogonality test. When we use survey inflation forecasts, we strongly reject the null hypothesis of orthogonality both in the case of the forecasts provided by the University of Michigan and the Fed of Philadelphia. At a 12 months horizon, the structural model is not rejected based on the same null hypothesis.

Table XI, *about here*

In what follows, we repeat the previous analysis for the monetary process. Let the prediction error be $u_{t+12} = \ln\left(\frac{M_{t+12}}{M_t}\right) - E_t\left[\ln\left(\frac{M_{t+12}}{M_t}\right)\right]$, we estimate and test whether u_{t+12} is orthogonal to lagged values of the explanatory variables, i.e. $E\left[u_{t+12} \otimes \phi\left(\frac{M_t}{M_{t-12}}\right)\right] = 0$. At a one month frequency, we reject the orthogonality hypothesis. At a one year frequency, the coefficients on lagged values of money are not, at the individual level, significantly different from zero, suggesting that the model does a reasonably good job in capturing the dynamics of the money supply at a one year frequency. As we increase the horizon, the performance along this dimension improves. In the limit, the model implied unconditional expected value of money growth rate is 6.81 with respect to an empirical value of 6.26. When we run a test of the null hypothesis that the unconditional moment is correctly specified, we cannot reject hypothesis. However, the higher the frequency, the more stgrongly we can reject the null hypothesis that the prediction residuals are orthogonal to past information, suggesting that at high frequency the model is missing a relevant explanatory variable.

Table XII, *about here*

[26]Neither The University of Michigan nor the Federal Reserve Bank of Philadelphia offer inflation forecasts at a one month frequency.

6.5 Inflation Risk Premium

Over the entire sample, the average risk premium on the inflation rate is 0.55% at a 10 years horizon and it ranges over time between 0.20% and 1.25%. The average term structure of the inflation risk premium figure, calculated over the entire sample, is presented in Panel B of Figure 5. The term structure is upward sloping.

Figure 5, *about here*

The explanation for such a pattern of the risk premium may be twofold. First, inflation is more difficult to predict over the more distant future. In the short run inflation is very much influenced by the short-term history of monetary policy. In the longer run, a larger number of factors, such as changes in the monetary policy, may be responsible for a higher extent of uncertainty and thus for a higher premium on nominal bonds.

Second, long term bonds are more sensitive to the changes in inflation than short term bonds. The higher duration of long term bonds amplifies the price impact of the inflation due to its correlation with nominal interest rates. Therefore, long nominal bond investors will require a higher risk premium.

Can we statistically reject the null hypothesis that the risk premium is time varying? The local volatility of the risk factors has been specified as a shifted square root process, $\sqrt{\sigma_{0v}^2 + \sigma_{1v}^2 v_t}$. Time variation in the risk premia arises from the time varying conditional volatility of the risk factors. If σ_{1v} were equal to zero, then the risk premium would be constant. The parameter σ_{1v} is statistically different from zero, thus suggesting that the inflation risk premium in the nominal term structure is time-varying. The overall extent of time variation is shown in Figure 5, while Figure 6 presents the evolution of the average risk premium in 6 subperiods of equal length of 6 years and 10 months. We find that the general pattern of the risk premium remains relatively stable over periods. The most important exception is the subperiod July 1980 to April 1987 during which the slope of the term structure is substantially higher than in other subperiods. It should be noticed that this subperiod is also the one with the highest volatility for the inflation rate and a relatively high average level of inflation. The Fed changed the monetary target twice in 1979 and 1982[27]. The high average level and slope of the term structure of the risk premium is thus consistent with the belief that the inflation risk premium is related to the stability and credibility of the monetary policy.

Figure 6, *about here*

The dynamics of the inflation risk premia is of particular interest. Time variation of the inflation risk premium is depicted on the Panel A of Figure 5. In periods of high nominal interest rates and inflation, such as during the 1982 recession, the drop in real interest rates is correlated with a substantial increase in the inflation risk premium. This result is of interest

[27] In 1979 the Fed changed its target from Federal Funds Rate to Money Growth and Nonborrowed Reserves. In 1982 the Fed decided to target Borrowed Reserves instead.

for macroeconomics reasons, as expectations on the inflation rates are often extrapolated from nominal interest rates and this procedure requires the knowledge of the risk premium on the inflation rate. Moreover, the result is useful also for capital budgeting reasons as we find that the inflation risk-adjustment for the cost of capital shows substantial time variation and it is sensitive to the inflation regime. The correlation between US economic growth, proxied by the GDP growth rate and the value of the inflation risk premium varies from -0.25 to -0.21 for different maturities.

Figure 7 shows the evolution of the total inflation risk premium in both the time and the maturity domain. Computing the correlation matrix of the estimated time-varying variables of the model, we find that there is a high and positive correlation between expected inflation, nominal rates and the inflation risk premium, see Table X. The correlation between real rates and inflation risk premium is substantially smaller. Since low levels of the real rate are usually a characteristics of recessionary periods, one may conclude that the inflation risk premium is higher in recessions.

Figure 7, *about here*

7 Treasury Inflation-Indexed Securities (TIIS)

On January 29, 1997, the Treasury held the first auction of Treasury Inflation Indexed Securities (TIIS)[28]. These securities have also become known as TIPS, which is an acronym for Treasury Inflation Protected Securities. One of the arguments used by the Treasury to motivate the creation of this new security was the reduction in the cost of borrowing. Clearly, the argument implicitly assumes that investors demand a significant inflation risk premium. Since the payoff of these securities is indexed to the inflation rate, the spread between nominal Treasury bonds and TIIS may provide additional useful information to identify more precisely the inflation risk premium, at least in the last four years of the dataset.

In what follows, since there are no academic studies on U.S. TIIS, we discuss the main characteristics of this market. Then, we report the evidence on the inflation risk premium based on the term structure of TIIS.

The structure of TIIS offering is the same as of nominal Treasury securities. They are distributed by the Treasury via an auction procedure and then traded in the secondary market. Auction bidding is in terms of real yields. During the auction the Treasury determines the coupon rate. The coupon payment is paid semi-annually on the security's inflation-adjusted principal amount. Therefore, the structure of TIIS is different from floating rate instruments in the sense that the coupon amounts change with the inflation, while the coupon rate remains the same. The inflation-adjusted principal amount is derived by multiplying the face amount by the ratio of the consumer price index on a specific day over to the level of the index on the original issue date. Consider, for instance, a 3.5% TIIS with the principal of

[28]Before 1997, no inflation-linked risk free instrument existed in the US, unlike other countries like UK, Israel and France.

$1000. If the inflation is 1% in six months, then the principal is adjusted upwards to 1010 and the semi-annual coupon payment is 3.5/2 percent of the inflation-adjusted principal[29][30].

Inflation-linked securities in all countries have an indexation lag which is determined by the technical difficulties in calculating an inflation index, mostly due to the time needed to collect the data, as well as to the need to know the amount of accrued interest for each market transaction. In the UK the indexation lag is about 8 months. For TIIS the indexation lag is about 3 months. Thus, the CPI-U index value, which the Treasury uses for indexation, is the CPI-U for the third preceding calendar month. No inflation shocks occurring in the previous three months of a coupon payment are reflected in the coupon payment.

The taxation of TIIS is controversial. The interest payments are taxed at the income tax rate as in the case of nominal securities. However, inflation-adjustments to the principal amount are also taxable as income, even if they are intended to protect the investor from inflation. This limits the ability of TIIS to protect against inflation and make them a more appropriate investment for tax-deferred accounts. The different tax treatment of nominal Treasury bonds and TIIS has direct implications for any exercise intended to extrapolate information about the real interest rate and the expected inflation from the yield spread between nominal and real securities. Without an adjustment for difference in taxation, one would overestimate the level of real rates in the economy and consequently underestimate either the expected inflation rate or the size of the inflation risk premium. The size of the tax adjustment can be quite large and is analyzed in the following section.

By the end of the year 2000, there are 8 TIIS issues outstanding: one 5 year note, four 10 year notes and two 30 year notes. Each issue is about $14 to $16 billion in size. The total inflation-adjusted face value of TIIS is $126.3 billion[31]. The market activity in inflation-linked securities remains moderate when compared to nominal Treasury securities. According to the Federal Reserve Bulletin by Dupont and Sack (1999), the daily trading volume in TIIS over the second quarter of 1999 averaged about 1.7 percent of TIIS outstanding compared with about 5.0 percent for nominal Treasury securities. The bid-offer spreads on these securities are typically from 1.6 to 6.3 basis point, which is much wider than the bid-offer spreads of nominal Treasury securities but much narrower than the spreads for corporate bonds.

[29] The Treasury uses non-seasonally adjusted US City Average All Items Consumer Price Index for all Urban Customers (CPI-U) as a reference inflation index. This index is published monthly by the Bureau of Labor Statistics.

[30] For trading purposes, the accrued interest needs to be computed on a daily basis at any intermediate date between two consecutive price index readings. The Treasury uses straight linear interpolation to compute the principal amount of an inflation-index security at any intermediate date between two consecutive CPI-U reading. This procedure allows the principal to be adjusted for inflation on a daily basis and therefore the accrued interest is known for trading purposes.

[31] Source: Monthly Statement of Public Debt available from Treasury web-site www.publicdebt.treas.gov

7.1 TIIS Dataset

To estimate the inflation risk premium implied by TIIS we use a dataset which contains market prices of inflation-linked securities from January 1997 to December 2000. The market for index-linked securities is still very young and illiquid and the number of securities used in the estimation increases from one 10 year note at the beginning of 1997 to 8 securities of 5, 10 and 30 years of maturity at the end of the year 2000.

From a first glance to the TIIS data, we notice that the term structure of real interest rates implied by TIIS is very flat with respect to the nominal term structure. The average level of the real term structure in our sample is 3.7 percent.

In principle, one could use the TIIS real yield to maturity as an approximation for the level of the real interest rate. This would be correct if it were indeed the case that the cash flows from TIIS were shielded from inflation. Unfortunately, this is not the case. Investors are required to pay taxes on the (nominal) appreciation of the face value of their TIIS. Since the income tax rate is 40%, this effect is potentially very large and it can generate a large bias in the estimate of the real interest rate. In order to have a feeling about the magnitude of the distortion, let Δ_r be the real interest rate bias due to ignoring the tax implications of the TIIS. Δ_r is given implicitly by the solution of the following equation:

$$\sum_{m=1}^{M} \frac{C\left(1-\tau\right)-\tau\pi}{\left(1+r\right)^{t_m}}+\frac{1}{\left(1+r\right)^{t_m}}=\sum_{m=1}^{M} \frac{C\left(1-\tau\right)}{\left(1+r+\Delta_r\right)^{t_m}}+\frac{1}{\left(1+r+\Delta_r\right)^{t_m}}$$

where M is the number of coupons, τ is the marginal tax rate, π is a constant inflation rate, r is the real discount rate. If the marginal income tax rate is 40% and the expected inflation is 3% the value of Δ_r is 1.2%. The bias can be large indeed.

The market for index linked securities has not been studied to the same extent of the nominal bond market. Thus, it is difficult to say what is the tax rate of a marginal investor in index linked bonds. Green and Odegaard (1997) shows that after the 1986 Tax Act the tax effects on the market for nominal US Treasury securities are very small. However, the composition of participants in the TIIS market differs significantly from the composition in the nominal Treasury markets. Table XIII summarizes the Distribution of Auction Award data available from the US Treasury and gives an idea of the differences in the existing clienteles in both markets[32].

Table XIII, *about here*

Table XIII shows that the typical allocation of index-linked bonds is significantly different from the one of auctions for fixed coupon or nominal securities. Primary dealers receive only 43% of the total amount of TIIS placed in auctions, versus 82% of the nominal Treasury

[32] Source: Department of Treasury, Quarterly Refunding Charts

Notes. At the same time, the percentage allocated to tax paying investors is substantially higher. This may suggest that the marginal tax rate relevant for pricing purposes in this market is different from the marginal tax rate in the nominal Treasury market.

We re-run the estimation of the structural model, this time using both quoted prices of nominal Treasury bonds and TIIS, accounting for the tax implications in similar way to Green and Odegaard (1997). The parameter values of the model are very close to the ones obtained using only the nominal term structure. The implied marginal tax rate for TIIS is found to be 21%. We find that for short maturities, i.e. less than one year, the inflation risk premium is very close to the one obtained using only nominal Treasury bonds. However, for a horizon between 3 and 7 years, the average risk premium is about 7 basis points higher.

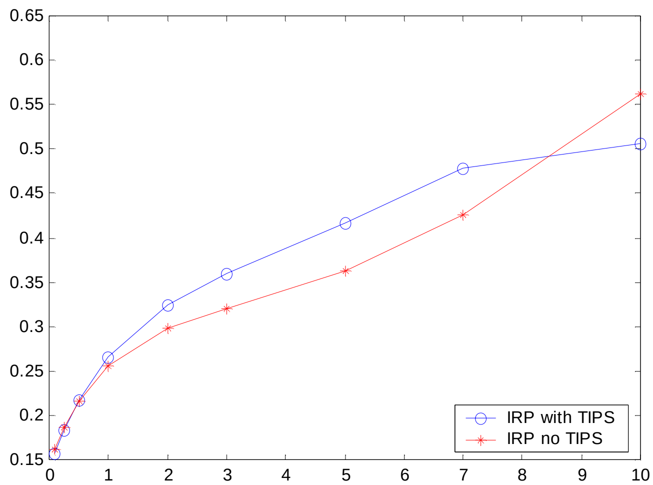

Figure 1: **The Inflation Risk Premium using TIPS**
This figure shows the estimated term structure of risk premia using only nominal Treasury bonds and using both nominal Treasury bonds and TIIS.

We also investigate the goodness of fit of the model with respect to TIIS prices. This may serve as an additional indication of the quality of the specification of the model. We find that the mean absolute pricing error is 23.8 basis points, slightly larger than the errors on nominal Treasury bonds but still small with respect to the standard deviation of the yields to maturity.

`Table` XIV, *about here*

8 The Term Premium and the Expectation Hypothesis

One of the most debated and studied financial relationships is the expectation hypothesis of interest rates. The economic motivation for such an interest is clear. If the expectation hypothesis were correct, at least in a statistical sense, one could use implied forward rates to obtain a good proxy for the expected future spot rate. Thus, it may be natural to ask under what conditions the expectation hypothesis of interest rates holds in our structural monetary economy.

The expectation hypothesis of the term structure states that agents' expectations about realization of future interest rates are fully embodied in the present level of interest rates, and therefore bond prices. In its pure form, i.e. without any allowance for the existence of a risk premium, it is alternatively formulated saying that (1) forward rates are unbiased predictors of futures rates (unbiased expectations hypothesis: U-EH), (2) the instantaneous expected return on bonds of any maturity is equal to the instantaneous interest rate (local expectations hypothesis: L-EH) or (3) the expected return from rolling over a money market account up to a certain maturity is equal to the return that can be obtained by purchasing an equal maturity zero coupon bond (returns to maturity expectations hypothesis: RTM - EH)[33].

If interest rates were deterministic, the three versions of the EH would produce the same zero coupon bond prices, for a given set of expectations about the future level of interest rates. However, under stochastic interest rates, it is well known[34] that the three formulations mentioned above cannot be simultaneously consistent with the absence of arbitrage. Since bond prices are convex functions of the interest rate, by Jensen's inequality the three different versions produce different term structures. However, it is frequently claimed that, for reasonable values of interest rate volatility, the adjustment due to Jensen's inequality is quite modest in quantitative terms[35] and the three versions of the theory are very similar[36].

Traditional tests of the expectations hypothesis are usually based on the U-EH. Although Cox, Ingersoll and Ross (1981) claim that this specification is incompatible with any continuous time rational-expectation economy, McCulloch (1993) shows an example of a homoskedastic stochastic endowment economy that supports bond prices satisfying the U-EH in its pure form (no term premium).

Let us then turn to compute the forward premium for the monetary equilibrium and

[33] There is a fourth version of the expectation hypothesis, namely the yield to maturity expectation hypothesis (YTM - EH) stating that the expected yield from rolling over instantaneously a money market account up to a certain maturity is equal to the one obtained by purchasing an equal maturity zero coupon. It can be easily shown that this formulation is exactly equivalent to the Unbiased Expectation Hypothesis, as discussed in Cox, Ingersoll and Ross(1981).

[34] See Cox, Ingersoll and Ross (1981) for a detailed discussion of the alternative specifications of the expectation hypothesis.

[35] This is not true for very long maturities, where the effect of Jensen's inequaliy can be quite substantial, as documented in Brown and Schaefer (1997).

[36] J. Campbell (1986) shows how the expectation hypothesis can be obtained as a log-linear approximation of the bond price process.

discuss the conditions under which the U-EH holds. Let the forward interest rate at time t for an instantaneous forward contract beginning at time $T = t+\tau$ be $f(t,T)$. The instantaneous forward rate is equal to $-\frac{\partial}{\partial T} ln\, B_t^{\tau}$, so that

$$f(t,T) = -\frac{A'(\tau)}{A(\tau)} + b_v'(\tau)v_t + \sum_i b_{z^i}'(\tau)z_t^i \tag{14}$$

From which, the following Proposition follows:

Proposition 4 (The Unbiased Expectation Hypothesis) *The matching maturity forward rate is a conditionally biased estimator of the expected future spot interest rate. The forward premium is time varying and it depends linearly on the level of the underlying pricing factors as follows:*

$$\begin{aligned} & f(t,T) - E_t\left[r_T\right] = \\ & \left[A_0 - \frac{A'(\tau)}{A(\tau)} + \frac{1}{2}\frac{(\Theta_1^v)^2+2\Theta_0^v\Theta_2^v}{\Theta_2^v}\frac{\theta}{k}(e^{-k\tau}-1) + \frac{1}{2}\sum_{i=1}^n \frac{(\Theta_1^{z^i})^2+2\Theta_0^{z^i}\Theta_2^{z^i}}{\Theta_2^{z^i}}\frac{\zeta^i}{\xi^i}(e^{-\xi^i\tau}-1)\right] \\ & + \left[b_v'(\tau) + \Theta_0^v e^{-k\tau}\right] v_t + \sum_{i=1}^n \left[b_{z^i}'(\tau) + \Theta_0^{z^i} e^{-\xi^i\tau}\right] z_t^i \end{aligned} \tag{15}$$

It can be noticed that the sign of the term premium can be strictly positive or negative, depending on the value assumed by the state variables. This depends on the stochastic volatility structure of the pricing factors, that makes the term premium time varying. Moreover, the presence of multiple factors can accommodate for different possible shapes of the term premium.

In what follows, we would like to study the issue of the time variation of the forward risk premium. We ask the following question: "If we generate term structure data using the structural model and run Campbell-Shiller type of regressions, do we find the same pattern in the slope coefficients?" If this were the case, the model would be describing an economy with the same empirical characteristics found in the expectation hypothesis literature. This would be very interesting indeed since the structural model could then provide an economic explanation of the empirical results found in the literature.

Campbell-Shiller regress the change in the constant time-of-maturity yield onto the current slope of the yield curve. Let R_t^{τ} be the yield on a Treasury bond with maturity $t+\tau$:

$$R_{t+m}^{n-m} - R_t^n = \alpha + \beta \underbrace{\left[\left(\frac{m}{n-m}\right)(R_t^n - R_t^m)\right]}_{S_t^{n,m}} + \varepsilon_t$$

The expectations hypothesis requires that $\beta = 1$. The regression coefficient is given by

$$\beta = \frac{cov\,(x,y)}{var\,(x)} = \frac{\left(\frac{m}{n-m}\right) cov\left(R_{t+m}^{n-m} - R_t^n, R_t^n - R_t^m\right)}{\left(\frac{m}{n-m}\right)^2 var\left(R_t^n - R_t^m\right)}$$

Since in our structural model the yields curve is affine in the states, so that

$$R_t^\tau = -\frac{\ln B(t,\tau)}{\tau} = -\frac{\ln A(\tau)}{\tau} + \frac{b_v(\tau)}{\tau} v_t + \sum_{i=1}^{n} \frac{b_{z^i}(\tau)}{\tau} z_t^i,$$

we can solve in closed-form for the regression coefficients. Let,

$$R_{t+m}^{n-m} - R_t^n = a^m(n,m) + \left[\frac{b_v(n-m)}{n-m} v_{t+m} - \frac{b_v(n)}{n} v_t\right] + \sum_{i=1}^{n} \left[\frac{b_{z^i}(n-m)}{n-m} z_{t+m}^i - \frac{b_{z^i}(n)}{n} z_t^i\right]$$

Defining $b^0(n,m) = \frac{b(n)}{n} - \frac{b(m)}{m}$, the denominator of the regression coefficient is equal to

$$var(R_t^n - R_t^m) = \left[b_v^0(n,m)\right]^2 Var(v_t) + \sum_{i=1}^{n} \left[b_{z^i}^0(n,m)\right]^2 Var\left(z_t^i\right)$$

The numerator of the regression coefficient is equal to[37]

$$cov\left(R_{t+m}^{n-m} - R_t^n, R_t^n - R_t^m\right) =$$

$$= \left[\frac{b_v(n-m)}{n-m} b_v^0(n,m)\right] cov(v_{t+m}, v_t) - \left[\frac{b_v(n)}{n} b_v^0(n,m)\right] var(v_t)$$

$$+ \sum_{i=1}^{n} \left[\frac{b_{z^i}(n-m)}{n-m} b_{z^i}^0(n,m)\right] cov\left(z_{t+m}^i, z_t^i\right) - \sum_{i=1}^{n} \left[\frac{b_{z^i}(n)}{n} b_{z^i}^0(n,m)\right] var\left(z_t^i\right)$$

[37] The unconditional moments of the factors are given by $E(v_t) = -\frac{\theta}{k}$, $E(z_t) = -\frac{\zeta}{\xi}$, $Var(v_t) = \frac{\sigma_{1v}^2 \theta}{2k^2} - \frac{\sigma_{0v}^2}{2k}$, $Var(z_t) = \sigma_z^2 \frac{\zeta}{2\xi^2}$, $E\left(v_t^2\right) = \frac{\sigma_{1v}^2 \theta}{2k^2} - \frac{\sigma_{0v}^2}{2k} + \frac{\theta^2}{k^2}$, and $E\left(z_t^2\right) = \sigma_z^2 \frac{\zeta}{2\xi^2} + \frac{\zeta^2}{\xi^2}$. Similarly, the two covariance terms can be calculated as

$$\begin{aligned}
cov(z_{t+m}, z_t) &= E\left[z_{t+m} - E(z_{t+m})(z_t - E(z_t))\right] = E\left[z_{t+m} z_t\right] - \left[E(z_t)\right]^2 \\
E\left[z_{t+m} z_t\right] &= E\left[E_t(z_{t+m}) z_t\right] = E\left(\left[-\frac{\zeta}{\xi} + e^{\xi m}\left(z_t + \frac{\zeta}{\xi}\right)\right] z_t\right) \\
&= e^{\xi m} E\left[z_t^2\right] + \left[\frac{\zeta}{\xi}\left(e^{\xi m} - 1\right)\right] E(z_t) \\
&= e^{\xi m}\left[\sigma_z^2 \frac{\zeta}{2\xi^2} + \frac{\zeta^2}{\xi^2}\right] - \left[\frac{\zeta}{\xi}\left(e^{\xi m} - 1\right)\right]\frac{\zeta}{\xi} \\
cov(z_{t+m}, z_t) &= E\left[z_{t+m} z_t\right] - \left[E(z_t)\right]^2 \\
&= e^{\xi m}\left[\sigma_z^2 \frac{\zeta}{2\xi^2} + \frac{\zeta^2}{\xi^2}\right] - \left[\frac{\zeta}{\xi}\left(e^{\xi m} - 1\right)\right]\frac{\zeta}{\xi} - \frac{\zeta^2}{\xi^2} \\
&= e^{\xi m} \sigma_z^2 \frac{\zeta}{2\xi^2}
\end{aligned}$$

Similarly, for the nominal factor we obtain:

$$\begin{aligned}
cov(v_{t+m}, v_t) &= E\left[v_{t+m} - E(v_{t+m})(v_t - E(v_t))\right] = E\left[v_{t+m} v_t\right] - \left[E(v_t)\right]^2 \\
&= e^{km}\left(\frac{\sigma_{1v}^2 \theta}{2k^2} - \frac{\sigma_{0v}^2}{2k}\right)
\end{aligned}$$

Let $\beta(\hat{\Theta})$ be the slope coefficient of the Campbell-Shiller regressions implied by the model, given the set of estimated structural parameters $\hat{\Theta}$, and let $\hat{\beta}$ be the empirical slope coefficient obtained by re-running the Campbell-Shiller regression on the extended dataset. Table XV summarizes the results. Both the absolute levels of the slope coefficient and their pattern as a function of the maturity closely mirror the results in Campbell and Shiller. The slope coefficient at a one year horizon implied by the structural model is -0.15 compared to a value of 0.11 obtained by Campbell and Shiller. As the horizon increases, the slope coefficients decrease as in Cambpell and Shiller. At a 7 year horizon, the implied slope regression coefficient is -2.85 while the empirical Campbell-Shiller value is -3.11. We run Chi-square tests of the null hypothesis that $H0 : \beta(\hat{\Theta}) = 1$ and also that the two sets of coefficients are equal, i.e. $H0 : \beta(\hat{\Theta}) = \hat{\beta}_{cs}$. We find that the implied values of the Campbell-Shiller regression coefficients strongly reject the expectation hypothesis at any confidence level. Second, the implied slope coefficients $\beta(\hat{\Theta})$ are not significantly different from the one obtained by Campbell and Shiller, with an average p-value equal to 0.3885.

Table XV, *about here*

We think that the ability of the model to replicate the rejection of the expectation hypothesis is due to the fact that (a) the risk premium is time-varying and state-dependent and (b) the equilibrium price of risk implied by the model is not directly proportional to the local volatility of the factors v_t and z_t^i, as discussed in Section (2.1). Several traditional reduced-form models of the term structure assume that the market risk premium is proportional to the volatility of the latent factors. Duarte (2000), Dai and Singleton (2001), Backus, Telmer, and Wu (1999) show that such assumption is an important limiting features of these models.

Is the rejection of the expectation hypothesis due to the inflation risk premium? The question of matching maturities forward rates as conditionally unbiased predictors of future spot rates has been addressed, among others, by Fama (1976) and Fama and Bliss (1987). Stambaugh (1988) tests if forward excess returns are unbiased linear predictors of excess holding period returns. More recently, Beckaert, Hodrick and Marshall (1997) develop a small sample test statistics and find strong empirical evidence against a generalized version of the U-EH, allowing for a constant risk premium. They interpret this failure observing that since the US economy has been very rarely in a high inflation regime, a steep term structure, due to expectations of large inflation shocks, might then be considered excessive from an *ex-post* perspective. They fit and test a regime switching model of interest rates finding supporting evidence for this explanation.

In the framework of this paper, we can directly compare the part of the term premium that is constant with the part that generates deviations from the expectation hypothesis. Moreover, using the overidentifying restrictions given by the structural monetary model, we can identify the part that is generated purely by nominal shocks. Thus, in our framework we can go a step further with respect to the previous literature and ask whether the rejection of the expectation hypothesis is due to time variation in the risk premium on nominal (monetary) shocks or to time variation of the risk premium on real (technological) shocks. From equation (15), this can be explored by testing whether

- $H1 : b'_v(\tau) + \Theta_0^v \exp(-k\tau) = 0$
- $H2 : b'_{z_i}(\tau) + \Theta_0^{z_i} \exp(-\xi^i\tau) = 0, \qquad i = 1, 2$

If deviations from the expectation hypothesis are due mainly to nominal shocks, we should expect $H1$ to be rejected, but not $H2$. Alternatively, if the deviations are due to the productivity shocks we should expect the opposite to hold. If both the two null hypothesis are rejected, the model can be used to compare the economic size of the two components.

We construct the following Wald test for the two potential sources of violations:

$$W_i = T\left[g_i(\theta_T)\right]' \left\{ \left[\left.\frac{g_i(\theta)}{\partial\theta}\right|_{\theta=\theta_T} \right] \Sigma_i \left[\left.\frac{g_i(\theta)}{\partial\theta}\right|_{\theta=\theta_T} \right]' \right\}^{-1} \left[g_i(\theta_T)\right] \qquad i = 1, 2$$

with $g_1(\theta)$ and $g_2(\theta)$ being the two testable restrictions from $H1$ and $H2$ and Σ_i being a consistent estimator of the covariance matrix of the residuals. Under the null hypothesis, the test statistics W_i is asymptotically Chi-squared distributed.

Table XVI presents the empirical results.

Table XVI, *about here*

We find that the expectation hypothesis is rejected both because of time variation in the risk premium on the nominal (monetary) factor and also because of the time variation in the risk premium of the real (technological) shocks.

The p-value for the nominal factor is always less than 1% for any maturity with the exception of the overnight rate. This result confirms the conjecture by Beckaert, Hodrick and Marshall (1997) that the rejection of the EH is due to the existence of a time varying risk premium generated by inflation risk. However, the evidence also shows that the forward premium is time varying even abstracting from inflation risk. The p-values of the first real (technological) factor are smaller than 1% for all maturities. The time-variation of the inflation risk premium is an important but not unique reason for the rejection of the expectation hypothesis.

9 Conclusions

In the theoretical part of this paper we study a monetary economy with taxes on nominal profits. The monetary side is modeled in a way in which the monetary authority can adjust the rate of growth of the money supply both to nominal and to real shocks. The adjustment is defined in terms of deviations from long term objectives in terms of output growth and inflation. In this economy, due to the imperfect indexation mechanism of the fiscal system to inflation, nominal shocks can generate distortions on the real capital accumulation and economic agents demand a risk premium on the level of inflation. Moreover, the stochastic process for the nominal and real risk factors is such that this risk premium is potentially time-varying. We solve for the general equilibrium of the economy and obtain closed form solutions for the term structure of nominal, index-linked bonds and for the risk premium on the inflation rate.

In the empirical section of the paper, we estimate the structural parameters of the economy using a panel data on US Treasury bonds ranging from 1960 to 2000 with maturity from 1 months to 10 years. From the empirical analysis we learn the following:

(a) The Fisher hypothesis is strongly rejected by the data: the average risk premium on the inflation rate is 0.60%.

(b) The risk premium on the level of inflation is highly time varying, ranging from 0.20% to 1.60%, and it is positively correlated with the actual level and volatility of inflation.

(c) We check whether the kind of time variation in the risk premium can generate the same regression slope coefficients as the ones found by Campbell and Shiller. Based on the estimated parameters of the model, we find a very similar pattern, i.e. negative and downward sloping regression slope coefficients.

(d) Since the model can separately identify both nominal and real factors, we test the importance of the time-variation in the inflation risk premium in rejections of the expectation hypothesis of interest rates. We find that the extent of its time variation is a sufficient reason for rejecting the expectation hypothesis.

(e) We explore the extent to which a structural monetary model can price jointly both nominal Treasury and index-linked bonds. We find that the same three common factors pricing the nominal term structure also price the term structure of real bonds reasonably well. However, we also confirm previous empirical evidence that affine specification have difficulty matching the second moments of yield changes.

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A Technical Appendix

A.1 Dynamic Capital Accumulation Equation

The representative consumer owns the company and decides how much to invest or consume. The explicit cost of capital is zero in the sense that holding the capital does not imply any cash outflows as it would be in the case of borrowed capital. However, there are two types of costs associated with the production technology, depreciation and variable costs. Both of them are deductable for tax purposes.
The capital gains tax is levied on the increase in the total nominal value of capital. If inflation does not change, than the capital gains tax is zero. Therefore, written in discrete time, the capital accumulation is

$$\begin{aligned} K_{t+h} &= K_t + K_t 1'(\mathsf{Y}_{t+h} - \mathsf{Y}_t) - \underbrace{\lambda_m K_t h}_{\text{depreciation}} - \underbrace{\lambda_s K_t 1'(\mathsf{Y}_{t+h} - \mathsf{Y}_t)}_{\text{variable production cost}} \\ & -\tau_{pr} \underbrace{\left(K_t 1'(\mathsf{Y}_{t+h} - \mathsf{Y}_t) - \frac{p_t}{p_{t+h}} \lambda_m K_t h - \frac{p_t}{p_{t+h}} \lambda_s K_t 1'(\mathsf{Y}_{t+h} - \mathsf{Y}_t) \right)}_{\text{taxable base: expenses deducted at historic cost}} - \underbrace{\tau_{cg} \left(\frac{p_{t+h} - p_t}{p_{t+h}} \right)}_{\text{capital gains tax}} K_t - C_t h - m_t^d h \end{aligned}$$

In continuous time capital, with $h \to dt$ and $(\mathsf{Y}_{t+h} - \mathsf{Y}_t) \to d\mathsf{Y}_t$, the accumulation equation becomes

$$\frac{dK_t}{K_t} = \left[dY_t (1 - \tau_{pr})(1 - \lambda_s) - \lambda_m (1 - \tau_{pr}) dt - \tau_{pr} \lambda_s COV_t \left(\frac{dp_t}{p_t}, dY_t \right) dt - \tau_{cg} \frac{dp_t}{p_t} + \tau_{cg} \sigma_P^2 (\cdot) dt - \frac{C_t}{K_t} dt - \frac{m_t^d}{K_t} dt \right] \tag{A.1}$$

A.2 Consumer Optimization Problem

Let us define the problem

$$\max_{\{C_t, M_t^d\}} E_0 \left[\int_0^\infty e^{-\rho t} \left[\ln(C_t) + \gamma \ln\left(M_t^d\right) \right] dt \right]$$

subject to the budget constraint (A.1).
In order to simplify the notation, let us define the following parameters:

$$\begin{array}{cc} A = -\lambda_m (1 - \tau_{pr}) & A_{py} = -\tau_{pr} \lambda_s \\ A_z = (1 - \lambda_s)(1 - \tau_{pr}) \mu_{y^i} & A_c = A_m = 1 \\ A_p^\mu = -\tau_{cg} & B_z = \sigma_{y^i} \\ A_p^\sigma = \tau_{cg} & B_p = -\tau_{cg} \end{array} \tag{A.2}$$

Then the capital accumulation equation (A.1) may be written in the following form.

$$\frac{dK_t}{K_t} = \left[A + A_z z_t + A_p^\mu \mu_p(\cdot) + A_p^\sigma \sigma_p^2(\cdot) + A_{py} cov \left(\frac{dp_t}{p_t}, dY_t^i \right) - A_c \frac{C_t}{K_t} - A_m \frac{m_t}{K_t} \right] dt \tag{A.3}$$

$$+ \left[B_z \sqrt{z_t^i} dW_t^{y^i} + B_p \sigma_p(\cdot) dW_t^p \right] \tag{A.4}$$

Where z_t^i are underlying real factors driving the productivity of capital

$$\begin{aligned} dy_t^i &= \mu_{y^i} z_t^i dt + \sigma_{y^i} \sqrt{z_t^i} dW_t^{y^i} \\ dz_t^i &= (\xi^i z_t^i + \zeta^i) dt + \sigma_{z^i} \sqrt{z_t^i} dW_t^{z^i} \end{aligned}$$

Let us assume that there exists an equilibrium price process that takes the form

$$\frac{dp_t^*}{p_t^*} = \mu_{p^*}(\cdot) dt + \sigma_{p^*}(\cdot) dW_t^{p^*}$$

We will later varify that this is indeed the case and solve for the market clearing functional values of μ_{p^*} and σ_{p^*}. The monetary supply policy is

$$\begin{aligned} \frac{dM_t^s}{M_t^s} &= v_t dt + q_1 \left(\frac{dK_t^*}{K_t^*} - \bar{\kappa} \right) dt + q_2 \left(\frac{dp_t^*}{p_t^*} - \bar{\pi} \right) + \sqrt{\sigma_{0M}^2 + \sigma_{1M}^2 v_t} dW_t^M \\ dv_t &= (k v_t + \theta) dt + \sqrt{\sigma_{0v}^2 + \sigma_{1v}^2 v_t} dW_t^v \end{aligned}$$

In equilibrium, there must exist a value function $J(t, K_t, z_t, v_t)$ and control variables $\{C_t, M_t^d\}$ such that the following Benvenieste-Scheinkman condition is satisfied

$$-\frac{\partial}{\partial t}J(\cdot) = \max_{\{C_t, M_t^d\}} \left[e^{-\rho t}\ln(C_t) + e^{-\rho t}\gamma \ln(M_t^d) + \mathcal{A}J(\cdot)\right]$$

where $\mathcal{A}J$ is the differential operator applied to the function $J(\cdot)$. Let us consider the following guess for $J(\cdot)$

$$\left[\frac{1}{\rho}e^{-\rho t}\right]^{-1} J(t, K_t, z_t^i, v_t) = P + \left[Q\ln(\rho K_t) + R_{z^i} z_t^i + R_v v_t\right]$$

$$-\frac{\partial}{\partial t}J(\cdot) = e^{-\rho t}\left(P + \left[Q\ln(\rho K_t) + R_{z^i} z_t^i + R_v v_t\right]\right)$$

Let us compute the differential $\mathcal{A}J(\cdot)$:

$$\begin{aligned}
\mathcal{A}J(t, K_t, z_t, v_t) &= \frac{\partial J}{\partial K}\mu_K + \frac{\partial J}{\partial z_t^i}\mu_{z_t^i} + \frac{\partial J}{\partial v_t}\mu_{v_t} + \frac{\partial^2 J}{\partial K^2}\sigma_K^2 + \frac{\partial^2 J}{\partial \left(z_t^i\right)^2}\sigma_{z_t^i}^2 + \frac{\partial^2 J}{\partial v_t^2}\sigma_{v_t}^2 + \frac{\partial^2 J}{\partial K \partial z_t^i} cov\left(K_t, z_t\right) + \frac{\partial^2 J}{\partial K \partial v_t} cov\left(K_t, v_t\right) \\
&= Q\frac{1}{K_t}\mu_K + \frac{1}{2}\left(-Q\frac{1}{K^2}\right)\sigma_K^2 + R_{z^i}\mu_{z^i} + R_v \mu_v \\
&= Q\left[A + A_z z_t + A_p^{\mu}\mu_p\left(\cdot\right) + A_p^{\sigma}\sigma_p^2\left(\cdot\right) + A_{pk} cov\left(\frac{dp_t}{p_t}, \frac{dK_t}{K_t}\right) - A_c\frac{C_t}{K_t} - A_m\frac{m_t}{K_t}\right] + +R_{z^i}\left(\xi^i z_t^i + \zeta^i\right) \\
&\quad + R_v\left(kv_t + \theta\right) - \frac{Q}{2}\left(B_z^2 z_t^i dt + B_p^2\sigma_p\left(\cdot\right)dt + COV_t\left[B_z\sqrt{z_t^i}dW_t^{y^i}, B_p\sigma_p\left(\cdot\right)dW_t^p\right]\right)
\end{aligned}$$

The first order conditions are:

$$\begin{aligned}
\left[C_t\right] &: \quad e^{-\rho t}\frac{1}{C_t} - A_c\left[\frac{1}{\rho}e^{-\rho t}\right]Q\frac{1}{K_t} = 0 \\
\left[M_t^d\right] &: \quad e^{-\rho t}\frac{\gamma}{M_t^d} - A_m\left[\frac{1}{\rho}e^{-\rho t}\right]Q\frac{1}{K_t} = 0
\end{aligned}$$

From which we obtain that the consumption and real money holdings are linear functions of total capital K_t:

$$\begin{aligned}
C_t &= A_c\frac{\rho}{Q}K_t \\
M_t^d &= A_m\frac{\gamma\rho}{Q}K_t, \ m_t = A_m\frac{\gamma\rho}{Q}dK_t
\end{aligned}$$

Let us solve for Q. Substitute the optimal policy functions in the Benvenieste-Scheinkman conditions

$$-\frac{\partial}{\partial t}J(\cdot) = \max_{(C_t, M_t^d\}} \left[e^{-\rho t}\ln(C_t) + e^{-\rho t}\gamma\ln(M_t^d)\right] + \mathcal{A}J(\cdot)$$

$$\begin{aligned}
&e^{-\rho t}\left[P + Q\ln(\rho K_t) + R_{z^i} z_t^i + R_v v_t\right] = e^{-\rho t}\ln\left[\frac{\rho}{Q}K_t\right] + \gamma e^{-\rho t}\ln\left[\frac{\gamma\rho}{Q}K_t\right] \\
&+\left[\frac{1}{\rho}e^{-\rho t}\right]\left(\begin{array}{c} Q\left[A + A_z z_t + A_p^{\mu}\mu_p\left(\cdot\right) + A_p^{\sigma}\sigma_p^2\left(\cdot\right) + A_{py} cov\left(\frac{dp_t}{p_t}, dy_t^i\right) - A_c\frac{\rho}{Q} - A_m\frac{\gamma\rho}{Q}\right] + R_{z^i}\left(\xi^i z_t^i + \zeta^i\right) + R_v\left(kv_t + \theta\right) \\ -\frac{Q}{2}\left(B_z^2 z_t^i + B_p^2\sigma_p^2\left(\cdot\right) + COV_t\left[B_z\sqrt{z_t^i}dW_t^{y^i}, B_p\sigma_p\left(\cdot\right)dW_t^p\right]\right) \end{array}\right)
\end{aligned}$$

The parameter values for $[P, Q, R_{z^i}, R_v]$ can be solved by matching the coefficient of the state variables [constant, $\ln(K)$, z_t^i, v_t] of the Benvenieste-Scheinkman optimality condition. One can notice that a solution exists only if $\mu_p\left(\cdot\right)$, $\sigma_p^2\left(\cdot\right)$ and $COV_t[B_z\sqrt{z_t^i}dW_t^{y^i}$, $B_p\sigma_p\left(\cdot\right)dW_t^p]$ are affine functions of the underlying factors. We will use this property later to solve for the equilibrium value of dp/p.

Let us first focus on Q. Matching the coefficients of $\ln(K)$:

$$e^{-\rho t}Q = e^{-\rho t} + \gamma e^{-\rho t}$$

we obtain $Q = (1+\gamma)$. It follows that the optimal policy functions are

$$\begin{aligned} C_t &= A_c \tfrac{\rho}{(1+\gamma)} K_t \\ M_t^d &= A_m \tfrac{\gamma\rho}{(1+\gamma)} K_t \end{aligned} \tag{A.5}$$

In order to solve for the other parameters and verify the guess for the indirect utility function, we need to solve for the equilibrium price process $\frac{dp^*}{p^*}$. This can be obtained from the market clearing condition for monetary holdings

$$p_t^* M_t^{*d} = M_t^s \tag{MCC}$$

Assuming that markets cleared at time $t = 0$, i.e. $p_0^* M_0^d = M_0^s$, using Ito's Lemma, the previous market clearing condition is equivalent to:

$$M_t^d dp_t^* + p_t dM_t^d + COV_t\left(dp_t^*, dM_t^d\right) = dM_t^s$$

Equivalently, dividing by M_t^d, and substituting the equilibrium value of dM_t^d/M_t^d, from (A.5)

$$\frac{dp_t^*}{p_t^*} + \frac{dK_t^*}{K_t^*} = \frac{dM_t^s}{M_t^s} - COV_t\left(\frac{dp_t^*}{p_t}, \frac{dK_t^*}{K_t^*}\right)$$

$$\begin{aligned} \frac{dp_t^*}{p_t^*} &= \frac{dM_t^s}{M_t^s} - \frac{dK_t^*}{K_t^*} - COV_t\left(\frac{dp_t^*}{p_t}, \frac{dK_t^*}{K_t^*}\right) = \\ & v_t dt + q_1 \frac{dK_t^*}{K_t^*} + q_2 \frac{dp_t^*}{p_t^*} + \sqrt{\sigma_{0M}^2 + \sigma_{1M}^2 v_t} dW_t^M - \frac{dK_t^*}{K_t^*} - COV_t\left(\frac{dp_t^*}{p_t}, \frac{dK_t^*}{K_t^*}\right) - (q_1\bar{\kappa} + q_2\bar{\pi})dt \end{aligned} \tag{A.6}$$

$$\begin{aligned} \frac{dK_t^*}{K_t^*} &= \left[A + A_z z_t + A_p^\mu \mu_p(\cdot) + A_p^\sigma \sigma_p^2(\cdot) + A_{py} cov\left(\frac{dp_t}{p_t}, dy_t^i\right) - \rho \frac{A_c + A_m\gamma}{(1+\gamma)}\right] dt && \text{(A.7)} \\ &+ \left[B_z \sqrt{z_t^i} dW_t^{y^i} + B_p \sigma_p(\cdot) dW_t^p\right] && \text{(A.8)} \\ &\equiv \mu_K^* dt + \sigma_K^* dW_t^K \end{aligned}$$

Equations (A.6) and (A.7) are a system of SDE that jointly define the market clearing conditions for money and the equilibrium process for capital respectively. Since, the uncertainty in both equations is driven by the same basis of Brownian motions $\left[W_t^{y^i}, W_t^M\right]$, we can solve this system of equation by considering the drifts and diffusion coefficients and have an exact representation expressed in terms of the underlying vector of factors $\left[z_t^i, v_t\right]$. The solutions are obtained by subtitutions[38].

$$\begin{aligned} \mu_K^* &= \frac{A - \frac{A_p^\mu(q_1\bar{\kappa}+q_2\bar{\pi})}{(1-q_2)} - \rho\frac{A_c+A_m\gamma}{(1+\gamma)} + A_z z_t + \frac{A_p^\mu}{(1-q_2)} v_t + COV_t\left(\frac{dp_t^*}{p_t}, \frac{dK_t^*}{K_t^*}\right)\left(A_{pk} - \frac{A_p^\mu}{(1-q_2)}\right) + A_p^\sigma \sigma_p^2(\cdot)}{1 + \frac{A_p^\mu(1-q_1)}{1-q_2}} \\ \mu_p^* &= \frac{1}{(1-q_2)} v_t + \frac{(q_1-1)}{(1-q_2)} \mu_K^* - \frac{(q_1\bar{\kappa}+q_2\bar{\pi})}{(1-q_2)} - \frac{COV_t\left(\frac{dp_t^*}{p_t}, \frac{dK_t^*}{K_t^*}\right)}{(1-q_2)} \\ COV_t\left(\frac{dp_t^*}{p_t^*}, \frac{dK_t^*}{K_t^*}\right) &= \left[\frac{1}{(1-q_2) + B_p(1-q_1)}\right]^2 \left[(q_1-1)(1-q_2) B_z^2 z_t^i + B_p\left(\sigma_{0M}^2 + \sigma_{1M}^2 v_t\right)\right] \\ A_{py} cov\left(\frac{dp_t}{p_t}, dy_t^i\right) &= \frac{(q_1-1)(1-q_2) B_z}{(1-q_2) + B_p(1-q_1)} \sigma_{y^i} z_t^i dW_t^{y^i} \\ \sigma_K^* dW_t^K &= \frac{1}{(1-q_2) + B_p(1-q_1)} \left[(1-q_2) B_z \sqrt{z_t^i} dW_t^{y^i} + B_p \sqrt{\sigma_{0M}^2 + \sigma_{1M}^2 v_t} dW_t^M\right] \\ \sigma_p^*(\cdot) dW_t^{P^*} &= \frac{1}{(1-q_2) + B_p(1-q_1)} \left[(q_1-1) B_z \sqrt{z_t^i} dW_t^{y^i} + \sqrt{\sigma_{0M}^2 + \sigma_{1M}^2 v_t} dW_t^M\right] \end{aligned}$$

[38] The exact derivation procedure is omitted for reasons of space. It is available from the authors upon request.

After we have expressions for equilibrium prices processes we can verify the original guess for the value function. Substitute the optimal demand functions into Bellman-Hamilton-Jacobi equation

$$-\frac{\partial}{\partial t}J(K_t, z_t, v_t) = \max_{\{C_t, M_t^d\}}\left[e^{-\rho t}\ln(C_t) + e^{-\rho t}\gamma\ln(M_t^d) + \mathcal{A}J(K_t, z_t, v_t)\right]$$

Substituting the equilibrium value of the drift and volatility of the price process and matching the coefficients of $\{const., \ln K_t,\ z_t^i,\ v_t\}$, it is possible to obtain a linear system of four equations in four unknown $\{P, Q, R_i, R_v\}$ whose solution are independent of the log-level of capital $\ln K_t$ and of the state variables $\{\mathsf{z}_t, v_t\}$. The guess can be considered verified.
Q.E.D.

Lemma 1 (Feynman-Kac Theorem for Unbounded Functions, Karatsas-Shreve)
Consider the diffusion process

$$dx_t = \mu(t, x_t)dt + \sigma(t, x_t)dW_t \tag{A.9}$$

and the function $f(t, x_t)$. The following Lemma, by Karatsas and Shreve, generalizes the standard Feynman-Kac theorem to unbounded functions $f(t, x_t)$ and non stationary processes x_t:
[A1] Let the parameters governing the vector diffusion process $\boldsymbol{x_t}$ be continuous and twice continuously differentiable functions in $\boldsymbol{x_t}$ and $\boldsymbol{dx_t}$ have a weak solution $\boldsymbol{x_t} \in R^k_{++}$ unique in probability law.
Let $f(t, \boldsymbol{x_t})$ be bounded below: $f(\boldsymbol{x_t}) \geq 0 \quad \forall \boldsymbol{x_t} \in R^k_{++}$
[A2] Let $\phi(t, \boldsymbol{x_t}) : [0, T] \times R^k_{++} \to R^1$ be continuous and of class $C^{1,2}$ on $[0, T] \times R^k_{++}$.
[A3] The function $\phi(t, \boldsymbol{x_t})$ be such that:

$$\max_{0\leq t\leq t+s}|\phi(t, x)| \leq \mathsf{M}\,(1 + \|x\|^{2\mu}\,) \tag{A.10}$$

for some constant $M > 0$ and $\mu \geq 1$.
If $\phi(t, \boldsymbol{x_t})$ satisfies the problem:

$$-\frac{\partial\phi(t, \boldsymbol{x_t})}{\partial t} = \mathcal{A}\phi(t, \boldsymbol{x_t}) \tag{A.11}$$

$$s.t. \qquad \phi(T, \boldsymbol{x_T}) = f(T, \boldsymbol{x_T}) \tag{A.12}$$

where $A\phi$ is the second order differential operator applied to the function ϕ, then $\phi(t, \boldsymbol{x_t})$ admits the stochastic representation:

$$\phi(t, \boldsymbol{x_t}) = E_{t, \boldsymbol{x_t}} f(T, \boldsymbol{x_T}) \tag{A.13}$$

A.3 The Term Structure of Nominal Interest Rates

Let $\kappa_t^* = (1+\gamma)\ln K_t^* + \rho t$. Recalling that the equilibrium diffusion process for dK_t^*/K_t^* is

$$\frac{dK_t^*}{K_t^*} = \mu_K^* dt + \sigma_K^* dW_t^K$$

we can use Ito's Lemma to compute $d\kappa_t^* \equiv d\ln K_t^* = \mu_{\kappa^*}dt + \sigma_{\kappa^*}dW_t^{\kappa^*}$

$$d\kappa_t^* = \left[(1+\gamma)\left[\mu_K^* - \frac{1}{2}Trace(\boldsymbol{\sigma}_K^*\boldsymbol{\sigma}_K^{*\prime})\right] + \rho\right]dt + (1+\gamma)\boldsymbol{\sigma}_K^{*\prime}d\mathsf{W}_t^{K^*}$$

From the standard first order conditions of the representative agent, we know that the real price of a zero coupon bond with a nominal payoff is equal to a unit of the numeraire is equal, in equilibrium, to the conditional expected value of the product of the intertemporal marginal rate of substitution times the real payoff of the financial asset:

$$\begin{aligned}
\frac{1}{p_t^*}B_t^\tau &= E_t\left[e^{-\rho\tau}\frac{\exp\left(-\ln X_{t+\tau}^*\right)}{\exp\left(-\ln X_t^*\right)}\frac{1}{p_{t+\tau}^*}\right] \\
&= E_t\left[e^{-\rho\tau}\frac{\exp\left(-(1+\gamma)\ln K_{t+\tau}^*\right)\frac{1}{p_{t+\tau}}}{\exp\left(-(1+\gamma)\ln K_t^*\right)\frac{1}{p_t}}\right] \\
&= E_t\left[\frac{\exp\left(-\left((1+\gamma)\ln K_{t+\tau}^* + \rho(t+\tau)\right)\right)\frac{1}{p_{t+\tau}}}{\exp\left(-((1+\gamma)\ln K_t^* + \rho t)\right)\frac{1}{p_t}}\right] \\
&= \frac{1}{\exp\left(-\kappa_t^*\right)\frac{1}{p_t}}E_t\left[\exp\left(-\kappa_{t+\tau}^*\right)\frac{1}{p_{t+\tau}}\right]
\end{aligned} \tag{A.14}$$

Let $\phi(\kappa_t^*, p_t^*, v_t, z_t; \tau)$ be the solution of the stochastic problem:

$$\phi(\kappa_t^*, p_t^*, v_t, \mathsf{z}_t; \tau) = E_t\left[\exp\left(-\kappa_{t+\tau}^*\right)\frac{1}{p_{t+\tau}^*}\right] \tag{A.15}$$

Since the economy has a constant return to scale production process and logarithmic preferences, let us consider the following log-linear guess:

$$\phi(\kappa_t^*, p_t^*, v_t, \mathsf{z}_t; \tau) = \left[\exp\left(-\kappa_t^*\right)\frac{1}{p_t^*}\right] A(\tau)\exp\left[-b_v(\tau)v_t - \sum_{i=1}^{n} b_{z^i}(\tau)z_t^i\right] \tag{A.16}$$

If $2\theta \geq \sigma_v^2$, then zero is an unattainable boundary for v_t and p_t and $\phi(\kappa_t^*, p_t^*, v_t, z_t; \tau)$ satisfies assumption $[A2]$ and the polynomial growth condition $[A3]$ of Lemma 1. By the Yamada-Watanabe theorem, it is possible to show that on R_+^2 the vector diffusion process $[p_s, v_s]$ has a weak solution for any initial condition $[t; p_t, v_t] \in [0, T] \times R_{++}^2$ so that it satisfies condition $[A1]$ of Lemma 1. $\phi(\kappa_t^*, p_t^*, v_t, z_t; \tau)$ is bounded below and satisfies the regularity conditions of Lemma 1, from which we know that if $\phi(\kappa_t^*, p_t^*, v_t, z_t; \tau)$ is solution to the stochastic problem (A.14), then it must also be a solution of the following differential problem:

$$\begin{aligned} -\frac{d}{dt}\phi(\kappa_t^*, p_t^*, v_t, \mathsf{z}_t; \tau) &= \mathcal{A}\phi(\kappa_t^*, p_t^*, v_t, \mathsf{z}_t; \tau) \\ s.t. \quad \lim_{\tau\to 0}\phi(\kappa_t^*, p_t^*, v_t, \mathsf{z}_t; \tau) &= \left[\exp\left(-\kappa_t^*\right)\frac{1}{p_t^*}\right] \end{aligned} \tag{A.17}$$

The left hand side of (A.17) is:

$$\frac{d}{d\tau}\phi^* = \left[\frac{A'(\tau)}{A(\tau)} - b_v'(\tau)v_t - \sum_{i=1}^{n} b_{z^i}'(\tau)z_t^i\right]\phi^* \tag{A.18}$$

Applying Ito's Lemma, the right hand side of (A.17) is

$$\begin{aligned} \mathcal{A}\phi(\kappa_t, p_t, v_t, \mathsf{z}_t; \tau) = {} & \frac{\partial\phi}{\partial\kappa}\mu_{\kappa^*} + \frac{\partial\phi}{\partial p}\mu_{p^*} + \frac{\partial\phi}{\partial v}\mu_v + \sum_{i=1}^{n}\frac{\partial\phi}{\partial z^i}\mu_{z^i} + \\ & + \frac{1}{2}\left[\frac{\partial^2\phi}{\partial\kappa^2}\sigma_{\kappa^*}^2 + \frac{\partial^2\phi}{\partial p^2}\sigma_{p^*}^2 + \frac{\partial^2\phi}{\partial v^2}\left(\sigma_0^2 + \sigma_v^2 v_t\right) + \sum_{i=1}^{n}\frac{\partial^2\phi}{\partial z^{i2}}\sigma_{z^i}^2\right] \\ & + \left[2\frac{\partial^2\phi}{\partial\kappa\partial p}Cov(d\kappa^*, dp^*) + 2\frac{\partial^2\phi}{\partial\kappa\partial v}Cov(d\kappa^*, dv) + 2\sum_{i=1}^{n}\frac{\partial^2\phi}{\partial\kappa\partial z^i}Cov(d\kappa^*, dz^i)\right] \\ & + \left[2\frac{\partial^2\phi}{\partial p\partial v}Cov(dp^*, dv) + 2\sum_{i=1}^{n}\frac{\partial^2\phi}{\partial p\partial z^i}Cov(dp^*, dz^i) + 2\sum_{i=1}^{n}\frac{\partial^2\phi}{\partial z^i\partial v}Cov(dz^i, dv)\right] \end{aligned}$$

Observe that

$$\begin{gathered} \tfrac{\partial\phi}{\partial\kappa^*} = -\phi,\ \tfrac{\partial\phi}{\partial p^*} = -\tfrac{1}{p^*}\phi,\ \tfrac{\partial\phi}{\partial v_t} = -b_v(\tau)\phi,\ \tfrac{\partial\phi}{\partial z_t^i} = -b_{z^i}(\tau)\phi. \\ \tfrac{\partial^2\phi}{\partial^2\kappa^*} = \phi,\ \tfrac{\partial^2\phi}{\partial^2 p^*} = 2\tfrac{1}{p^2}\phi,\ \tfrac{\partial^2\phi}{\partial^2 v} = b_v^2(\tau)\phi,\ \tfrac{\partial^2\phi}{\partial^2 z^i} = b_{z^i}^2(\tau)\phi \\ \tfrac{\partial^2\phi}{\partial k\partial p} = \tfrac{1}{p}\phi,\ \tfrac{\partial^2\phi}{\partial k\partial v} = b_v(\tau)\phi,\ \tfrac{\partial^2\phi}{\partial k\partial z^i} = b_{z^i}(\tau)\phi,\ \tfrac{\partial^2\phi}{\partial p\partial v} = \tfrac{1}{p}b_v(\tau)\phi,\ \tfrac{\partial^2\phi}{\partial p\partial z^i} = \tfrac{1}{p}b_{z^i}(\tau)\phi. \end{gathered}$$

After substituting the variance and covariance terms and some algebraic manipulation (assuming that $\rho_{v,M} = 0$), it is possible to obtain that the infinitesimal generator $\mathcal{A}$ is an affine function of underlying state variables. $\left[v_t, z_t^i\right]$ of the form

$$\mathcal{A}\phi(\kappa_t, p_t, v_t, \mathsf{z}_t; \tau) = \eta_0 + \eta_v v_t + \sum_{i=1}^{n}\eta_z z_t^i \tag{A.19}$$

where the parameters η_0, η_v and η_{z^i} are not time-varying and depend only on the set of structural parameters of the model Ω and a set of functions $b_z(\tau)$ and $b_v(\tau)$ of the maturity of the bond. The exact functional forms of $b_v(\tau)$ and $b_z(\tau)$ are presented in section A.3.

$$\begin{aligned} \eta_0 &= A_0(\Omega) + A_v(\Omega)\, b_v(\tau) + A_z(\Omega)\, b_z(\tau) + B_v(\Omega)\, b_v^2(\tau) \\ \eta_v &= \Theta_0^v(\Omega) + \Theta_1^v(\Omega)\, b_v(\tau) + \Theta_2^v(\Omega)\, b_v^2(\tau) \\ \eta_z &= \Theta_0^{z^i}(\Omega) + \Theta_1^z(\Omega)\, b_z(\tau) + \Theta_2^z(\Omega)\, b_z^2(\tau) \end{aligned}$$

where the functions

$$\begin{array}{c} A_0\left(\Omega\right), A_v\left(\Omega\right), A_z\left(\Omega\right), B_v\left(\Omega\right) \\ \Theta_0^v\left(\Omega\right), \Theta_1^v\left(\Omega\right), \Theta_2^v\left(\Omega\right) \\ \Theta_0^{z^i}\left(\Omega\right), \Theta_1^z\left(\Omega\right), \Theta_2^z\left(\Omega\right) \end{array} \tag{A.20}$$

are known non-linear functions of structural parameters Ω of the models. They are obtained simply by rearranging the terms in the expression for the infinitesimal generator $\mathcal{A}$ to the affine form (A.19). In order to save space we opt not to present the exact functional forms here. They are available from the authors upon request.
Thus, the solution of the bond pricing equation can be obtained by matching the coefficients of the state variables in (A.18) to the ones in (A.19). The functions in (A.20) are known functions of structural parameters of the model. Therefore, the pricing equation is equivalent to solving the following system of ODE:

$$\frac{A'(\tau)}{A(\tau)} = A_0 + A_v b_v(\tau) + A_z b_z(\tau) + B_v b_v^2(\tau) \tag{A.21}$$

$$-b_v'(\tau) = \Theta_0^v + \Theta_1^v b_v(\tau) + \Theta_2^v b_v^2(\tau) \tag{A.22}$$

$$-b_{z^i}'(\tau) = \Theta_0^{z^i} + \Theta_1^{z^i} b_z(\tau) + \Theta_2^{z^i} b_z^2(\tau) \tag{A.23}$$

The solution to this sistem of ODEs is discussed in the next section. The nominal price of a nominal zero coupon bond B_t^τ, with time to maturity τ, is a log-linear function of the real productivity and nominal shocks z_t^i, v_t. The closed form solution is:

$$B_t^t(\kappa_t, p_t, v_t, \mathsf{z}_t; 0) = A(\tau;\Omega)\exp\left[-b_v(\tau;\Omega)v_t - \sum_{i=1}^{n} b_{z^i}(\tau;\Omega)z_t^i\right] \tag{A.24}$$

$$A(\tau) = \exp\left(A_0\left(\Omega\right)\tau\right) a_v(\tau;\Omega)c_v(\tau;\Omega)\prod_{i=1}^{n} a_{z^i}(\tau;\Omega)$$

where $A(\tau;\Omega), b_v(\tau;\Omega), b_{z^i}(\tau;;\Omega)$ are obtained as a solutions of (A.21-A.23). The general solution of this system of ODE's is given in the following Lemma

Lemma: The General Solution of the Pricing ODE
Consider the system of ODE's (A.21), (A.22) and (A.23), it can be immediately noticed that the ordinary differential equations for both the nominal and real factors (A.22) and (A.23) have the same general form. Moreover, it can be verified by direct substitution that they admit a solution of the following form:

$$b(\tau) = \frac{1}{2\Theta_2}\left(-\Theta_1 + \sqrt{D}\tan\left[\arctan\left(\frac{\Theta_1}{\sqrt{D}}\right) - \frac{1}{2}\tau\sqrt{D}\right]\right), \text{where } D = -\Theta_1^2 + 4\Theta_0\Theta_2 \tag{A.25}$$

Let us now turn to solve equation (A.21):

$$\frac{A'(\tau)}{A(\tau)} = A_0 + A_v b_v(\tau) + B_v b_v^2(\tau) + \sum_{i=1}^{n} A_{z^i} b_{z^i}(\tau)$$

and let us consider the following educated guess:

$$A(\tau) = \exp(a_0\tau)a_v(\tau)c_v(\tau)\prod_{i=1}^{n} a_{z^i}(\tau)$$

which implies

$$\frac{A'(\tau)}{A(\tau)} = \frac{d}{d\tau}\left[\ln A(\tau)\right] = a_0 + \frac{a_v'(\tau)}{a_v(\tau)} + \frac{c_v'(\tau)}{c_v(\tau)} + \sum_{i=1}^{n}\frac{a_{z^i}'(\tau)}{a_{z^i}(\tau)}$$

Thus, the pricing restriction is equivalent to:

$$a_0 + \frac{a_v'(\tau)}{a_v(\tau)} + \frac{c_v'(\tau)}{c_v(\tau)} + \sum_{i=1}^{n}\frac{a_{z^i}'(\tau)}{a_{z^i}(\tau)} = A_0 + A_v b_v(\tau) + B_v b_v^2(\tau) + \sum_{i=1}^{n} A_{z^i} b_{z^i}(\tau)$$

which can be solved by solving the following system of individual restrictions:

$$\begin{aligned} a_0 &= A_0; \quad \frac{a_v'(\tau)}{a_v(\tau)} = A_v b_v(\tau), \\ \frac{c_v'(\tau)}{c_v(\tau)} &= B_v b_v^2(\tau); \quad \frac{a_{z^i}'(\tau)}{a_{z^i}(\tau)} = A_{z^i} b_{z^i}(\tau) \end{aligned}$$

It can be verified by direct substitution that the solutions are:

$$a_v(\tau) = 2^{\frac{A_v}{\theta_2}} \exp(-\frac{A_v \tau \theta_1}{2\theta_2}) \left[\cos\left(\arctan\left(\frac{\Theta_1}{\sqrt{D}}\right) - \frac{1}{2}\tau\sqrt{D}\right)\right]^{\frac{A_v}{\theta_2}} \left(\frac{\theta_2\theta_2}{D}\right)^{\frac{A_v}{2\theta_2}} \text{where } D = -\Theta_1^2 + 4\Theta_0\Theta_2 \tag{A.26}$$

$$a_z(\tau) = 2^{\frac{A_x}{\theta_2}} \exp(-\frac{A_z \tau \theta_1}{2\theta_2}) \left[\cos\left(\arctan\left(\frac{\Theta_1}{\sqrt{D}}\right) - \frac{1}{2}\tau\sqrt{D}\right)\right]^{\frac{A_z}{\theta_2}} \left(\frac{\theta_2\theta_2}{D}\right)^{\frac{A_z}{2\theta_2}} \text{where } D = -\Theta_1^2 + 4\Theta_0\Theta_2 \tag{A.27}$$

$$\begin{aligned} c_v(\tau) \quad = \quad & \exp\left[\frac{1}{2\Theta_2}\left(B_v\Theta_2 + B_v\tau\Theta_2 - 2B_v\tau\Theta_2\Theta_2 - B_v\Theta_1 \log\left(1 + \frac{\Theta_2}{D}\right) - B_v\sqrt{D}\tan\left[\arctan\left(\frac{\Theta_1}{\sqrt{D}}\right) - \frac{1}{2}\tau\sqrt{D}\right]\right)\right] \\ & \left[\cos\left(\arctan\left(\frac{\Theta_1}{\sqrt{D}}\right) - \frac{1}{2}\tau\sqrt{D}\right)\right]^{-\frac{B_v\Theta_1}{\theta_2^2}} \text{where } D \text{ is} -\Theta_1^2 + 4\Theta_0\Theta_2 \end{aligned} \tag{A.28}$$

It is worth noticing that in order to avoid the existence of arbitrage, the previous solution should also be non-periodic. Let us define $D \equiv -\Theta_1^2 + 4\Theta_0\Theta_2$. A well known necessary and sufficient condition for the solution of a Riccati equation of the type $-b(\tau) = \Theta_0 + \Theta_1 b(\tau) + \Theta_2 b^2(\tau)$ to be non-periodic is that $D < 0$.

A.4 The Term Structure of Real Interest Rates

From the standard first order condition of the representative agent, we know that the real price of a zero coupon inflation-linked (real) bond is equal, in equilibrium, to the conditional expected value of the product of the intertemporal marginal rate of substitution. Let IL_t^τ be the price of the index-linked bond with τ years to maturity, then

$$\begin{aligned} B_t^\tau &= E_t\left[e^{-\rho\tau}\frac{\exp\left(-\ln X_{t+\tau}^*\right)}{\exp\left(-\ln X_t^*\right)}\right] \\ &= E_t\left[e^{-\rho\tau}\frac{\exp\left(-(1+\gamma)\ln K_{t+\tau}^*\right)}{\exp\left(-(1+\gamma)\ln K_t^*\right)}\right] \\ &= E_t\left[\frac{\exp\left(-\left((1+\gamma)\ln K_{t+\tau}^* + \rho(t+\tau)\right)\right)}{\exp\left(-((1+\gamma)\ln K_t^* + \rho t)\right)}\right] \\ &= \frac{1}{\exp\left(-\kappa_t^*\right)} E_t\left[\exp\left(-\kappa_{t+\tau}^*\right)\right] \end{aligned} \tag{A.29}$$

Let $\phi(\kappa_t^*, p_t^*, v_t, z_t; \tau)$ be the solution of the stochastic problem:

$$\phi(\kappa_t^*, p_t^*, v_t, \mathsf{z}_t; \tau) = E_t\left[\exp\left(-\kappa_{t+\tau}^*\right)\right] \tag{A.30}$$

Since the economy has a constant return to scale production process and logarithmic preferences, let us consider the following log-linear guess:

$$\phi(\kappa_t^*, p_t^*, v_t, \mathsf{z}_t; \tau) = \left[\exp\left(-\kappa_t^*\right)\right] A(\tau) \exp\left[-b_v(\tau)v_t - \sum_{i=1}^n b_{z^i}(\tau) z_t^i\right] \tag{A.31}$$

If $2\theta \geq \sigma_v^2$, then zero is an unattainable boundary for v_t and p_t and $\phi(\kappa_t^*, p_t^*, v_t, z_t; \tau)$ satisfies assumption $[A2]$ and the polynomial growth condition $[A3]$ of Lemma 1. By the Yamada-Watanabe theorem, it is possible to show that on R_+^2 the vector diffusion process $[p_s, v_s]$ has a weak solution for any initial condition $[t; p_t, v_t] \in [0, T] \times R_{++}^2$ so that it satisfies condition $[A1]$ of Lemma 1. $\phi(\kappa_t^*, p_t^*, v_t, z_t; \tau)$ is bounded below and satisfies the regularity conditions of Lemma 1, from which we know that if $\phi(\kappa_t^*, p_t^*, v_t, z_t; \tau)$ is solution to the stochastic problem (A.14), then it must also be a solution of the following differential problem:

$$\begin{aligned} -\frac{d}{dt}\phi(\kappa_t^*, p_t^*, v_t, \mathsf{z}_t; \tau) &= \mathcal{A}\phi(\kappa_t^*, p_t^*, v_t, \mathsf{z}_t; \tau) \\ s.t. \quad \lim_{\tau \to 0} \phi(\kappa_t^*, p_t^*, v_t, \mathsf{z}_t; \tau) &= \left[\exp\left(-\kappa_t^*\right)\right] \end{aligned} \tag{A.32}$$

The left hand side of (A.17) is:

$$\frac{d}{d\tau}\phi^* = \left[\frac{A'(\tau)}{A(\tau)} - b_v'(\tau)v_t - \sum_{i=1}^n b_{z^i}'(\tau) z_t^i\right]\phi^* \tag{A.33}$$

Applying Ito's Lemma, the right hand side of (A.17) it is possible to obtain (after some algebraic manipulation) that the infinitesimal generator $\mathcal{A}$ is an affine function of underlying state variables. $\left[v_t, z_t^i\right]$ of the form

$$\mathcal{A}\phi = \eta_0^{IL} + \eta_v^{IL} v_t + \sum_{i=1}^{n} \eta_z^{IL} z_t^i \tag{A.34}$$

where the parameters η_0, η_v and η_{z^i} are not time-varying and depend on the set of structural parameters of the model Ω and set of functions $b_z(\tau)$ and $b_v(\tau)$ of the maturity of the bond only. The exact functional form of $b_v(\tau)$ and $b_z(\tau)$ is presented in section A.3.
Equating LHS and RHS of equation (A.32) we obtain coefficients for the following system of ODE:

$$\frac{A'(\tau)}{A(\tau)} = A_0 + A_v b_v(\tau) + A_z b_z(\tau) + B_v b_v^2(\tau) \tag{A.35}$$

$$-b_v'(\tau) = \Theta_0^v + \Theta_1^v b_v(\tau) + \Theta_2^v b_v^2(\tau) \tag{A.36}$$

$$-b_{z^i}'(\tau) = \Theta_0^{z^i} + \Theta_1^{z^i} b_z(\tau) + \Theta_2^{z^i} b_z^2(\tau) \tag{A.37}$$

Solving the system of ODEs (generic solution is given by in section) we obtain equation (A.38) which is the analog of equation (A.24) for nominal term structure

$$B_t^t(\kappa_t, p_t, v_t, \mathsf{z}_t; 0) = A^{IL}(\tau;\Omega) \exp\left[-b_v^{IL}(\tau;\Omega) v_t - \sum_{i=1}^{n} b_{z^i}^{IL}(\tau;\Omega) z_t^i\right] \tag{A.38}$$

$$A(\tau;\Omega) = \exp\left(A_0^{IL}(\Omega)\,\tau\right) a_v^{IL}(\tau;\Omega) c_v^{IL}(\tau;\Omega) \prod_{i=1}^{n} a_{z^i}^{IL}(\tau;\Omega)$$

A.5 Inflation Risk Premium

From standard first order conditions, the price of a nominal bond is equal to

$$B_t^\tau = E_t\left[e^{-\rho\tau} \frac{\exp(-\kappa_{t+\tau}^*)}{\exp(-\kappa_t^*)} \frac{p_t^*}{p_{t+\tau}^*}\right]$$

Similarly, the price of index-linked bonds is equal to:

$$IL_t^\tau = E_t\left[e^{-\rho\tau} \frac{\exp(-\kappa_{t+\tau}^*)}{\exp(-\kappa_t^*)}\right]$$

Thus, expanding the pricing equation for the nominal bond into the expected values of the marginal rate of substitution and the expected value of the reciprocal of the inflation rate, the risk premium on the inflation rate can be obtained as:

$$COV_t\left[e^{-\rho\tau} \frac{\exp(-\kappa_{t+\tau}^*)}{\exp(-\kappa_t^*)}; \frac{p_t^*}{p_{t+\tau}^*}\right] = B_t^\tau - IL_t^\tau \times E_t\left[\frac{p_t^*}{p_{t+\tau}^*}\right] \tag{A.39}$$

From equation (A.39) the risk premium can be calculated as a difference between the price of nominal bond and the price of real bond adjusted for expected inflation. In yield terms, the risk premium can be expressed in terms of the difference between the yield on the nominal bond and the yield on the index-linked bond adjusted by inflation.
Proposition 3 offers closed form solutions for B_t^τ, while IL_t^τ was previously computed. Thus, in order to obtain the closed-form solution of the inflation risk premium, we just need to compute the expected value of the reciprocal of the price process.
Let $\phi(p_t^*, v_t, z_t; \tau)$ be the solution of the stochastic problem:

$$\phi(p_t^*, v_t, \mathsf{z}_t; \tau) = E_t\left[\frac{1}{p_{t+\tau}^*}\right] \tag{A.40}$$

Since the economy has a constant return to scale production process and logarithmic preferences, let us consider the following log-linear guess

$$\phi(p_t^*, v_t, \mathsf{z}_t; \tau) = \left[\frac{1}{p_t^*}\right] A(\tau) \exp\left[-b_v(\tau) v_t - \sum_{i=1}^{n} b_{z^i}(\tau) z_t^i\right] \tag{A.41}$$

If $2\theta \geq \sigma_v^2$, then zero is an unattainable boundary for v_t and p_t and $\phi(\kappa_t^*, p_t^*, v_t, z_t; \tau)$ satisfies assumption $[A2]$ and the polynomial growth condition $[A3]$ of Lemma 1. By the Yamada-Watanabe theorem, it is possible to show that on R_+^2 the vector diffusion process $[p_s, v_s]$ has a weak solution for any initial condition $[t; p_t, v_t] \in [0, T] \times R^2_{++}$ so that it satisfies condition $[A1]$ of Lemma 1. $\phi(\kappa^*_t, p^*_t, v_t, z_t; \tau)$ is bounded below and satisfies the regularity conditions of Lemma 1, from which we know that if $\phi(\kappa^*_t, p^*_t, v_t, z_t; \tau)$ is solution to the stochastic problem (A.40), then it must also be a solution of the following differential problem:

$$\begin{aligned} -\frac{d}{dt}\phi(\kappa^*_t, p^*_t, v_t, \mathsf{z}_t; \tau) &= \mathcal{A}\phi(\kappa^*_t, p^*_t, v_t, \mathsf{z}_t; \tau) \\ s.t. \quad \lim_{\tau \to 0} \phi(\kappa^*_t, p^*_t, v_t, \mathsf{z}_t; \tau) &= [\exp(-\kappa^*_t)] \end{aligned} \tag{A.42}$$

It is possible to select coefficient in the guess function (A.41) in order to solve problem (A.42) in exactly the same fashion as similar problems (A.17) and (A.32) were solved for nominal and real bond prices respectively. The final for of the reciprocal of inflation can be found as

$$E_t\left[\frac{p^*_t}{p^*_{t+\tau}}\right] = A^{INF}(\tau;\Omega)\exp\left[-b^{INF}_v(\tau;\Omega)v_t - \sum_{i=1}^{n} b^{INF}_{z^i}(\tau;\Omega) z^i_t\right] \tag{A.43}$$

$$A^{INF}(\tau;\Omega) = \exp\left(A^{INF}_0(\Omega)\,\tau\right) a^{INF}_v(\tau;\Omega) c^{INF}_v(\tau;\Omega) \prod_{i=1}^{n} a^{INF}_{z^i}(\tau;\Omega)$$

Hence, we can calculate the inflation risk premoium in the closed form using the equation (A.39).

B Tables and Figures

Table I: Summary Statistics

This table presents summary statistics of the dataset used in the estimation. It is based on 492 monthly observations from January 1960 to December 2000. *Inflat* stands for the observed inflation rate calculated as the 12 months percentage change in the CPI index. *M2 growth* stands for the observed 12 months percentage change in the M2 money stock. All the other values refer to yield of nominal bonds for different maturities.

	Nobs	Max	Mean	Median	Min	Std
yield(1m)	492	16.2100	5.8549	5.4715	1.6060	2.5939
yield(3m)	492	15.9990	6.1450	5.6365	2.1850	2.6667
yield(6m)	492	16.5110	6.3758	5.8270	2.4420	2.6873
yield(1y)	492	16.3450	6.5810	6.0670	2.5830	2.6591
yield(2y)	492	16.1450	6.8019	6.3885	2.8730	2.5894
yield(3y)	492	15.8250	6.9377	6.4965	3.1170	2.5357
yield(5y)	492	15.6960	7.1191	6.7446	3.4360	2.4877
yield(7y)	492	15.2830	7.2441	6.9249	3.6070	2.4662
yield(10y)	492	15.0650	7.3361	7.1046	3.7320	2.4276
Inflat	492	13.6210	4.3318	3.5232	0.6689	2.8603
M2 growth	492	13.7869	6.8173	7.2847	0.3163	2.9223

Table II: Correlation Matrix for Time Series Used in The Estimation

This table presents the correlation matrix of the dataset used in the estimation. It is based on 492 monthly observations from January 1960 to December 2000. *Inflat* stands for the observed inflation rate calculated as the 12 months percentage change in the CPI index. *M2 growth* stands for the observed 12 months percentage change in the M2 money stock. All the other values refer to yield of nominal bonds for different maturities.

	1m	3m	6m	1y	2y	3y	5y	7y	10y	Inflat	M2 growth
1m	1.000	0.994	0.986	0.972	0.946	0.924	0.891	0.863	0.842	0.704	0.188
3m		1.000	0.996	0.985	0.962	0.942	0.910	0.883	0.862	0.720	0.191
6m			1.000	0.995	0.976	0.957	0.927	0.901	0.880	0.726	0.204
1y				1.000	0.991	0.978	0.953	0.932	0.912	0.712	0.205
2y					1.000	0.997	0.983	0.968	0.953	0.679	0.208
3y						1.000	0.994	0.984	0.973	0.659	0.205
5y							1.000	0.997	0.991	0.634	0.196
7y								1.000	0.998	0.616	0.185
10y									1.000	0.609	0.170
Inflat										1.000	0.168
M2 growth											1.000

Table III: Model Selection: Test for the Number of Factors

This table reports the asymptotic p-values for Andrews and Ploberger (1994) test statistics for parameters not identified under the null hypothesis. The p-value are generated using B. Hansen (1996) method. The brief description of the test is as follows. Let the measurement equation be:

$$x_t = \mathcal{M}\left(\alpha_{1t}(\theta_1)'\theta_{1,0} + \alpha_{2t}(\theta_2)'\theta_{2,0}\right) + \varepsilon_t$$

Let us consider the test of the null hypothesis $H_o : \theta_{2,0} = 0$, i.e. the second real latent factor is statistically redundant. Under the null hypothesis θ_2 does not enter the regression and it is not identified. This test statistics takes explicitly into account this issue. The null hypothesis is rejected at the 5% confident interval if the p-value is smaller than 0.05. More generally, $H_o : \theta_{j,0} = 0$ has the interpretation of the j-th real factor being statistically redundant.

	Andrews and Ploberger (1994) $\exp[T_n]$
$H_o : \theta_{2,0} = 0$	<1%
$H_o : \theta_{3,0} = 0$	0.11

Table IV: Goodness of Fit by maturity

This table presents goodness of fit summary for the model. All errors are measured in basis points. The fitting error is defined as the difference between the model generated nominal spot rate and the observed nominal rate for the period January 1960 to December 2000. The average absolute pricing error is 16.9 basis points. In the corresponding maturity code "m"stands for month and "y"stands for year.

maturity	*Mean Absolute Error*	*Median Absolute Error*
N1m	21.7	13.4
N6m	13.7	9.3
N1y	17.0	12.2
N2y	8.5	6.3
N5y	16.6	14.3
N7y	24.4	20.3

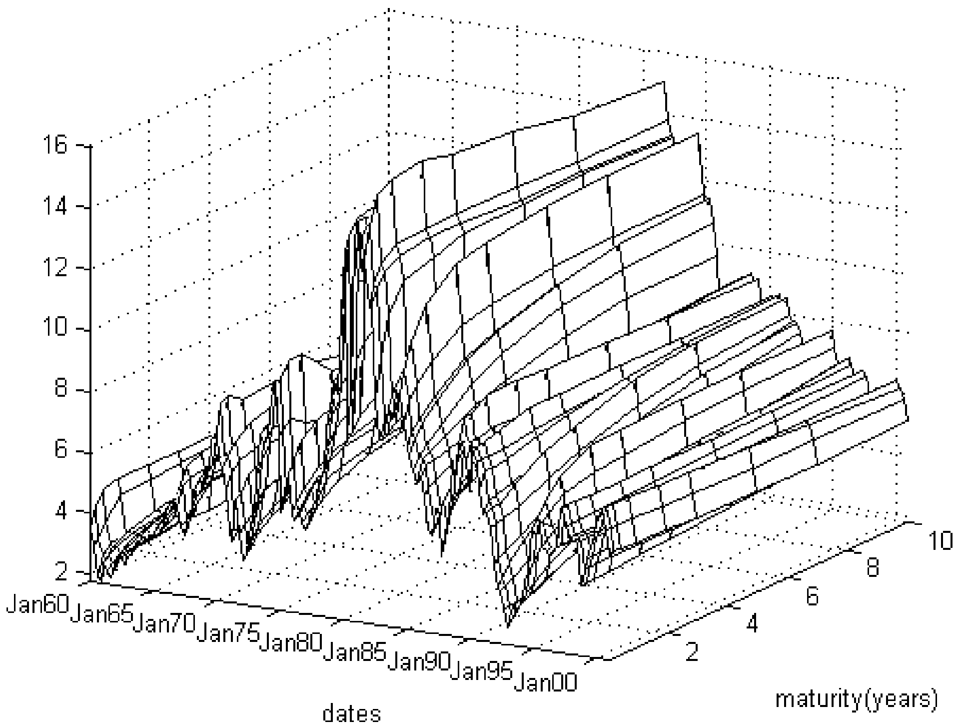

Figure 2: **The US Nominal Term Structure**

This figure presents the historical dynamics of the estimated yield curve. Each monthly yield curve is estimated on the basis of all end-of-day trading prices of Treasury bills, notes and bonds. The sample contains 444 monthly estimates of the yield curve for 9 different maturities ranging from 1 month to 10 years. The time span is from January 1960 to December 2000

Table V: **Parameter Estimates based on Maximum Likelihood Estimation**

This table presents the maximum likelihood parameter estimates and their standard errors. The estimation is based on 492 monthly observations from January 1960 to December 2000. The estimation is based on 9 maturities ranging from 1 month to 10 years. The estimated model has three factors. The two real factors follow $dz_t^i = \left(\xi^i z_t^i + \zeta^i\right) dt + \sigma_z^i \sqrt{z_t^i} dW_t^{z_i}$ where $i = 1, 2$. The capital depreciation rate is denoted with λ_m; the capital setup cost is λ_s; the capital gains and profits tax are τ_c and τ_p respectively. The Brownian motions $W_t^{y_i}$ and $W_t^{z_i}$ are correlated with correlation coefficient $\rho_{y_i z_i}$. The monetary authority sets the money supply M_t^s endogenously as $dM_t^s/M_t^s = v_t dt + q_1(\frac{dK_t^*}{K_t^*} - \bar{\kappa} dt) + q_2(\frac{dp_t^*}{p_t^*} - \bar{\pi} dt) + \sqrt{\sigma_{0M}^2 + \sigma_{1M}^2 v_t} dW_t^M$. Moreover, v_t follows the process $dv_t = (kv_t + \theta)dt + \sqrt{\sigma_{0v}^2 + \sigma_{1v}^2 v_t} dW_t^v$. In parenthesis we report the p-values of the Likelihood Ratio test.

λ_m	λ_s	τ_p	τ_c	γ	
1.135e-001	5.506e-002	2.171e-001	2.684e-001	4.589e-001	
(0.66)	(<1%)	(<1%)	(<1%)	(<1%)	
q_1	q_2	$\bar{\kappa}$	$\bar{\pi}$	ρ	
1.132e+000	-1.068e+000	3.416e-002	4.332e-002	7.461e-002	
(<1%)	(<1%)	(<1%)	(<1%)	(0.89)	
σ_{M_0}	σ_{M_1}	k	θ	σ_{v1}	σ_{v0}
1.975e+000	2.229e+000	-1.539e-001	1.415e+000	6.426e-002	1.443e-001
(<1%)	(<1%)	(<1%)	(<1%)	(<1%)	(0.02)
ζ_1	ξ_1	σ_{z_1}	μ_{y_1}	σ_{y_1}	$\rho_{z_1 y_1}$
1.270e+000	-3.645e-001	7.080e-002	6.765e-001	6.075e-001	-5.678e-001
(<1%)	(<1%)	(<1%)	(<1%)	(<1%)	(<1%)
ζ_2	ξ_2	σ_{z_2}	μ_{y_2}	σ_{y_2}	$\rho_{z_2 y_2}$
3.035e-001	-1.049e+000	2.194e-001	-1.054e+000	1.276e+000	-3.171e-001
(0.36)	(<1%)	(<1%)	(<1%)	(<1%)	(0.10)

Table VI: **The Effective Corporate Tax Rate**

Historical and recent data on profits and taxes for large corporations from CTJ/ITEP studies. The unit of measure are $-million unless otherwise indicated.

	Effective Tax Rate	*Annual Pretax US Profits*	*Annual Tax Breaks*	*% Below Statutory Rate*
1981-85	14.3%	$100,914	$31,990	-69%
After 1986 Tax Reform Act	26.5%	130,170	9,763	-22%
1996	22.9%	222,192	26,909	-35%
1997	22.3%	249,864	31,774	-36%
1998	20.1%	263,428	39,277	-43%

Table VII: **Volatility of Yields Fit**

Panel A. Term Structure of Yield Volatility

This table presents the goodness of fit for the volatility of yields differences. The conditional volatility is given by:

$$Var\left[y\left(\tau\right)\right] = \left[\frac{b_v\left(\tau\right)}{\tau}\right]^2 Var\left(v\right) + \sum_{i=1}^{n}\left[\frac{b_{z^i}\left(\tau\right)}{\tau}\right]^2 Var\left(z^i\right)$$

where v_t and z_t^i are stochastic factors and $b\left(\tau\right)$ are known function of maturity and structural parameters. "Model Vol" stands for the unconditional volatility of yields at different maturities. "Data Vol" are the sample unconditional volatilities of yields. χ^2-test is a test of the null hypothesis that the volatilities of yields implied by the model are equal to their sample counterpart. The unit of measure are percentage terms.

	1m	3m	6m	1y	2y	3y	5y	7y	10y
Model Vol	2.446	2.431	2.410	2.375	2.325	2.301	2.328	2.459	2.898
Data Vol	2.594	2.667	2.687	2.659	2.589	2.536	2.488	2.466	2.428
χ^2-test	0.646	0.484	0.410	0.376	0.385	0.422	0.567	0.985	0.038

Panel B. Term Structure of Volatility of Yield Changes

This table presents the goodness of fit for the volatility of yields differences. The volatility is given by:

$$Var\left[y_{t+\tau}\left(m\right) - y_t\left(m\right)\right] = \left[\frac{b_v\left(\tau\right)}{\tau}\right]^2 Var\left(v_{t+\tau} - v_t\right) + \sum_{i=1}^{n}\left[\frac{b_{z^i}\left(\tau\right)}{\tau}\right]^2 Var\left(z_{t+\tau}^i - z_t^i\right)$$

where v_t and z_t^i are stochastic factors and $b\left(\tau\right)$ are known function of maturity and structural parameters. "Model Vol" stands for the unconditional volatility of yield differences at different maturities. "Data Vol" are the sample unconditional volatilities of yield changes. χ^2-test is a test of the null hypothesis that the volatilities of changes in yields impled by the model are equal to their sample counterpart. The unit of measure are percentage terms.

	1m	3m	6m	1y	2y	3y	5y	7y	10y
Model Vol	0.466	0.404	0.368	0.334	0.302	0.287	0.277	0.285	0.325
Data Vol	0.657	0.575	0.559	0.541	0.486	0.444	0.389	0.352	0.328
χ^2-test	0.060	0.080	0.072	0.038	0.023	0.014	0.013	0.037	0.904

Table VIII: **Conditional Volatility Test: 1960-2000**

This table reports the result of a test that the conditional second moments of the model match their sample counterparts:

$$\left(y^n_{t+\Delta t} - E_t\left[y^n_{t+\Delta t}\right]\right)^2 = \alpha + \beta E_t\left[\left(y^n_{t+\Delta t} - E_t\left[y^n_{t+\Delta t}\right]\right)^2\right] + \varepsilon_t$$

$$H_0 : \alpha = 0 \quad \text{and} \quad H_0 : \beta = 1$$

In parenthesis we report the p-values of the test statistics.

$\Delta t = 1$ month

n	1m	3m	6m	1y	2y	3y	5y	7y	10y
α	-0.535	-0.420	-0.410	-0.378	-0.298	-0.236	-0.184	-0.135	-0.118
$H_0 : \alpha = 0$ $p{-}value$	(0.033)	(0.035)	(0.034)	(0.035)	(0.035)	(0.021)	(0.010)	(0.003)	(0.001)
β	3.853	2.991	2.878	2.670	2.131	1.728	1.339	1.032	0.900
$H_0 : \beta = 1$ $p{-}value$	(0.073)	(0.113)	(0.137)	(0.152)	(0.205)	(0.249)	(0.411)	(0.900)	(0.582)

$\Delta t = 6$ months

n	1m	3m	6m	1y	2y	3y	5y	7y	10y
α	-1.651	-1.256	-1.259	-1.111	-0.847	-0.709	-0.578	-0.443	-0.399
$H_0 : \alpha = 0$ $p{-}value$	(0.031)	(0.047)	(0.051)	(0.044)	(0.028)	(0.015)	(0.008)	(0.003)	(0.002)
β	2.449	1.930	1.930	1.741	1.356	1.124	0.889	0.692	0.607
$H_0 : \beta = 1$ $p{-}value$	(0.197)	(0.334)	(0.351)	(0.382)	(0.536)	(0.766)	(0.705)	(0.114)	(0.015)

$\Delta t = 1$ year

n	1m	3m	6m	1y	2y	3y	5y	7y	10y
α	-0.897	-0.631	-0.598	-0.495	-0.485	-0.470	-0.454	-0.389	-0.352
$H_0 : \alpha = 0$ $p{-}value$	(0.133)	(0.177)	(0.182)	(0.176)	(0.099)	(0.057)	(0.021)	(0.011)	(0.014)
β	1.436	1.125	1.104	0.986	0.849	0.744	0.627	0.519	0.453
$H_0 : \beta = 1$ $p{-}value$	(0.579)	(0.852)	(0.875)	(0.979)	(0.684)	(0.370)	(0.070)	(0.001)	(0.000)

Table IX: **Conditional Volatility Test: 1982-1998**

This table reports the result of a test that the conditional second moments of the model match their sample counterparts:

$$\left(y^n_{t+\Delta t} - E_t\left[y^n_{t+\Delta t}\right]\right)^2 = \alpha + \beta E_t\left[\left(y^n_{t+\Delta t} - E_t\left[y^n_{t+\Delta t}\right]\right)^2\right] + \varepsilon_t$$

$$H_0 : \alpha = 0 \quad \text{and} \quad H_0 : \beta = 1$$

In parenthesis we report the p-values of the test statistics.

$\Delta t = 1$ month

n	1m	3m	6m	1y	2y	3y	5y	7y	10y
α	-0.216	-0.332	-0.298	-0.221	-0.166	-0.147	-0.109	-0.085	-0.063
$\underset{p-value}{H_0 : \alpha = 0}$	(0.185)	(0.120)	(0.083)	(0.059)	(0.047)	(0.021)	(0.021)	(0.022)	(0.055)
β	1.875	1.864	1.615	1.322	1.118	1.035	0.856	0.754	0.678
$\underset{p-value}{H_0 : \beta = 1}$	(0.365)	(0.453)	(0.481)	(0.569)	(0.762)	(0.899)	(0.466)	(0.088)	(0.010)

$\Delta t = 6$ months

n	1m	3m	6m	1y	2y	3y	5y	7y	10y
α	-0.875	-0.854	-0.669	-0.586	-0.476	-0.465	-0.389	-0.352	-0.380
$\underset{p-value}{H_0 : \alpha = 0}$	(0.104)	(0.129)	(0.123)	(0.101)	(0.087)	(0.063)	(0.063)	(0.051)	(0.039)
β	1.255	1.091	0.882	0.813	0.723	0.703	0.601	0.545	0.548
$\underset{p-value}{H_0 : \beta = 1}$	(0.720)	(0.904)	(0.838)	(0.680)	(0.417)	(0.314)	(0.105)	(0.030)	(0.034)

$\Delta t = 1$ year

n	1m	3m	6m	1y	2y	3y	5y	7y	10y
α	-0.048	-0.137	-0.144	-0.180	-0.143	-0.174	-0.184	-0.222	-0.262
$\underset{p-value}{H_0 : \alpha = 0}$	(0.465)	(0.374)	(0.347)	(0.305)	(0.320)	(0.269)	(0.226)	(0.187)	(0.172)
β	0.509	0.434	0.407	0.438	0.421	0.422	0.389	0.381	0.379
$\underset{p-value}{H_0 : \beta = 1}$	(0.228)	(0.096)	(0.040)	(0.038)	(0.011)	(0.005)	(0.001)	(0.001)	(0.002)

Table X: Correlation Matrix for Model Generated Series

This table presents the correlation matrix for the estimated time series using the structural model. The first three variables are Nominal interest rates at maturities of one month, five years and ten years. The following three variables are Real interest rates for the same maturities. Rp1m, Rp5y and Rp10y indicate the inflation risk premium at a one month, 5 years and 10 years horizon respectively. The last variable, ModInf is the expected inflation rate estimated with the structural model.

	N1m	N5y	N10y	R1m	R5y	R10y	Rp1m	Rp5y	Rp10y	ModInf
N1m	1.000	0.890	0.827	-0.444	-0.502	-0.284	0.118	0.487	0.552	0.722
N5y		1.000	0.986	-0.706	-0.422	-0.101	0.481	0.750	0.802	0.644
N10y			1.000	-0.717	-0.370	-0.026	0.620	0.851	0.891	0.609
R1m				1.000	0.676	0.419	-0.514	-0.592	-0.623	-0.496
R5y					1.000	0.933	0.049	-0.146	-0.185	-0.649
R10y						1.000	0.356	0.203	0.170	-0.489
Rp1m							1.000	0.923	0.893	0.134
Rp5y								1.000	0.997	0.398
Rp10y									1.000	0.441
ModInf										1.000

Table XI: Orthogonality Tests of the Expected Inflation

The table presents the results of the orthogonality test for the inflation forecast. Let u_{t+1} be the prediction error for the inflation rate from the structural model and from (a) the Federal Reserve Bank of Philadelphia, (b) the survey data from The University of Michigan and (c) the random walk model. We study the extent to which the inflation forecasts are orthogonal to lagged explanatory variables by testing the null hypothesis $H_0 : \theta = 0$ in a GMM framework with the foll0wing moment restrictions

$$\begin{bmatrix} u_{t+1}(\theta) \\ u_{t+1}(\theta) \otimes [\phi(x_t)] \end{bmatrix}$$

with the unrestricted prediction errors defined as

$$u_{t+1} = \pi_{t+1} - E_t(\pi_{t+1} | I_t) - \theta' \phi(x_t)$$

We test the null hypothesis using the following statistics d_T

$$d_T = T \cdot \left[h_T(x_t, \theta(H_0))' W_T^{-1} h_T(x_t, \theta(H_0)) - h_T(x_t, \theta^*)' W_T^{-1} h_T(x_t, \theta^*) \right]$$

which is χ^2 distributed under the null hypothesis. The p-value associated to the d_T statistics and of the restricted parameters are in parenthesis. We consider the following lagged explanatory variables

$$\phi(x_t) = \begin{bmatrix} const \\ \pi_t \\ \pi_t^2 \end{bmatrix}$$

	Source of Forecasts					
	Model	*Random Walk*	*Model*	*Random Walk*	*Philadelphia*	*Michigan*
	Forecast Horizon					
	1 month		1 year			
	$t,t+1$	$t,t+1$	$t,t+12$	$t,t+12$	$t,t+12$	$t,t+12$
constant	0.346	0.143	0.215	0.423	-1.050	-1.844
	(0.319)	(0.000)	(0.414)	(0.243)	(0.005)	(0.000)
π	-0.455	-0.463	-0.309	0.054	0.194	0.779
	(0.108)	(0.000)	(0.281)	(0.428)	(0.160)	(0.000)
π^2	0.062	0.114	0.036	-0.022	-0.009	-0.052
	(0.010)	(0.234)	(0.157)	(0.131)	(0.340)	(0.025)
d_T	15.538	72.167	2.762	11.546	17.914	37.660
	(0.001)	(0.000)	(0.430)	(0.009)	(0.000)	(0.000)

Table XII: Orthogonality Tests of the Monetary Holdings

The table presents the results of the orthogonality tests of the prediction errors of monetary holding. We test the null hypothesis $H_0 : \theta = 0$ in the GMM framework with the following moment restrictions

$$\begin{bmatrix} u_{t+12}(\theta) \\ u_{t+12}(\theta) \otimes [\phi(x_t)] \end{bmatrix}$$

The prediction errors are defined as

$$u_{t+12} = \ln\left(\frac{M_{t+12}}{M_t}\right) - E_t\left[\ln\left(\frac{M_{t+12}}{M_t}\right)\right] - \theta'\phi(x_t)$$

We test the null hypothesis using the following statistics d_T

$$d_T = T \cdot \left[h_T(x_t, \theta(H_0))' W_T^{-1} h_T(x_t, \theta(H_0)) - h_T(x_t, \theta^*)' W_T^{-1} h_T(x_t, \theta^*)\right]$$

which is χ^2 distributed under the null hypothesis. The p-value of estimated parameters of the lagged inflation rate and of d_T-statistic are in parenthesis. We consider the following set of lagged explanatory variables:

$$\phi(x_t) = \begin{bmatrix} const \\ \ln\left(\frac{M_t}{M_{t+12}}\right) \\ \left[\ln\left(\frac{M_t}{M_{t+12}}\right)\right]^2 \end{bmatrix}$$

The last column shows the empirical and model-implied unconditional moment of the rate of growth of money. The last row shows the GMM Chi-square statistics, with p-values in parenthesis, of a test of the null hypothesis that the unconditional expected value mean of the model is equal to its sample counterpart.

	Forecast Horizon			*Unconditional Moment*
	1 month	1 year		
constant	0.136	1.908	*empirical value*	6.81
	(0.000)	(0.077)		
$\ln(\frac{M_t}{M_{t-12}})$	-0.212	-0.046	*model implied*	6.26
	(0.002)	(0.447)		
$\left[\ln(\frac{M_t}{M_{t-12}})\right]^2$	-0.031	-0.028		
	(0.280)	(0.104)		
d_T	64.907	21.192	d_T	2.07
	(0.000)	(0.000)		(0.15)

Table XIII: Inflation Linked and Fixed Rate Auction Distributions

This table summarizes the allocation of three inflation linked and corresponding fixed rate auctions in 1/99, 7/99 and 1/00. The following classification is applied, Primary dealer are main dealers in goverment securities selected by New York Fed. Financial institutions are nonprimary dealers, depository institutions and insurance companies. Inevstement Funds are investment managers, mutual funds and hedge funds. Others are individuals, non-financial companies and other financial companies

	Inflation Index Note	*Fixed Rate Note*
Primary Dealers	43%	82%
Investment Funds	20%	10%
Financial Intitutions	9%	3%
Pension Funds	3%	0%
Foreign	9%	1%
Other	20%	4%

Table XIV: **TIIS Pricing Errors**

This table presents goodness of fit summary for the TIPS data. All errors are measured in basis points. The fitting error is defined as the difference between the model generated real spot rate and the estimated real rate for the period from January 1997 to December 2000. The empirical real rates are estimated from Treasury Inflation Protected Securitity (TIPS) observed yields. In the corresponding maturity code “m”stands for month and “y”stands for year

	Mean Absolute Error	*Median Absolute Error*
1m	28.8	28.1
3m	27.9	27.1
6m	26.4	26.8
1y	23.8	23.8
2y	20.6	18.3
3y	18.8	18.4
5y	20.7	18.5
7y	28.4	25.1
10y	39.5	40.6

Table XV: Campbell and Shiller Regressions

This table reports the Campbell and Shiller regressions. The main regression equation is

$$R_{t+m}^{n-m} - R_t^n = \alpha + \beta \left(\frac{m}{n-m} \right) (R_t^n - R_t^m) + \varepsilon_t$$

where R_t^n is the yield of a bonds with maturity n at time t. The expectation hypothesis implies that that the coefficient β is equal to 1. The value of m is taken to be one month. Panel A presents the results of Cambell and Shiller regressions based on the updated dataset. Panel B presents the values of the same β coefficient *implied* by the structural model at the estimated values of the structural parameters. Closed-form solution for the coefficient is given in the appendix. Standard errors are given in parenthesis.

Panel A. Empirical Campbell and Shiller Coefficients, $\hat{\beta}_{cs}$

3m	6m	1y	2y	3y	5y	7y	10y
0.216	-0.33	-0.93	-1.22	-1.54	-2.21	-2.90	-3.67
(0.171)	(0.27)	(0.42)	(0.59)	(0.69)	(0.86)	(0.98)	(1.23)

Panel B. Model-Implied Campbell and Shiller Coefficients, $\beta(\hat{\Theta})$

3m	6m	1y	2y	3y	5y	7y	10y
0.106	0.0183	-0.1564	-0.5138	-0.8937	-1.7677	-2.8485	-4.9151
$p-values$ of $H0 : \beta(\hat{\Theta}) = 1$							
0.0000	0.0001	0.0029	0.0051	0.0030	0.0006	0.0000	0.0000
$p-values$ of $H0 : \beta(\hat{\Theta}) = \hat{\beta}_{cs}$							
0.2600	0.0985	0.0327	0.1157	0.1745	0.3035	0.4790	0.1557

Table XVI: Unbiased Expectation Hypothesis Test

This table presents the results of the Unbiased Expectation Hypothesis (U-EH) tests. U-EH claims that the matching maturity forward rate is a conditionally unbiased estimator of the expected future spot interest rate. We directly compare the part of the term premium that is constant with the part that generates deviations from the expectation hypothesis. Moreover, using the overidentifying restrictions of the structural monetary model, we can identify the part that is generated purely by nominal shocks. Two null hypothesis are tested: H1 : $b'_v(\tau) + \Theta_0^v \exp(-k\tau) = 0$ and H2 : $b'_{z_i}(\tau) + \Theta_0^{z_i} \exp(-\xi^i \tau) = 0$. Under $H1$, the nominal factor does not cause deviations from U-EH. Under $H2$, the real factor does not cause deviations from the U-EH Hypothesis. The table presents the results for different maturities in the term structure. The table reports the values of the Wald statistics and associated p-value.

	Nominal Factor	*First Real Factor*	*Second Real Factor*
0m	5.74e-013	0.00e+000	7.86e-012
	100.0%	100.0%	100.0%
1m	1.46e+001	3.00e+001	9.56e-001
	<1%	<1%	32.8%
12m	1.46e+001	3.01e+001	9.56e-001
	<1%	<1%	32.8%
3y	1.46e+001	3.04e+001	9.55e-001
	<1%	<1%	32.9%
5y	1.46e+001	3.06e+001	9.51e-001
	<1%	<1%	32.9%
10y	1.45e+001	3.11e+001	9.36e-001
	<1%	<1%	33.3%

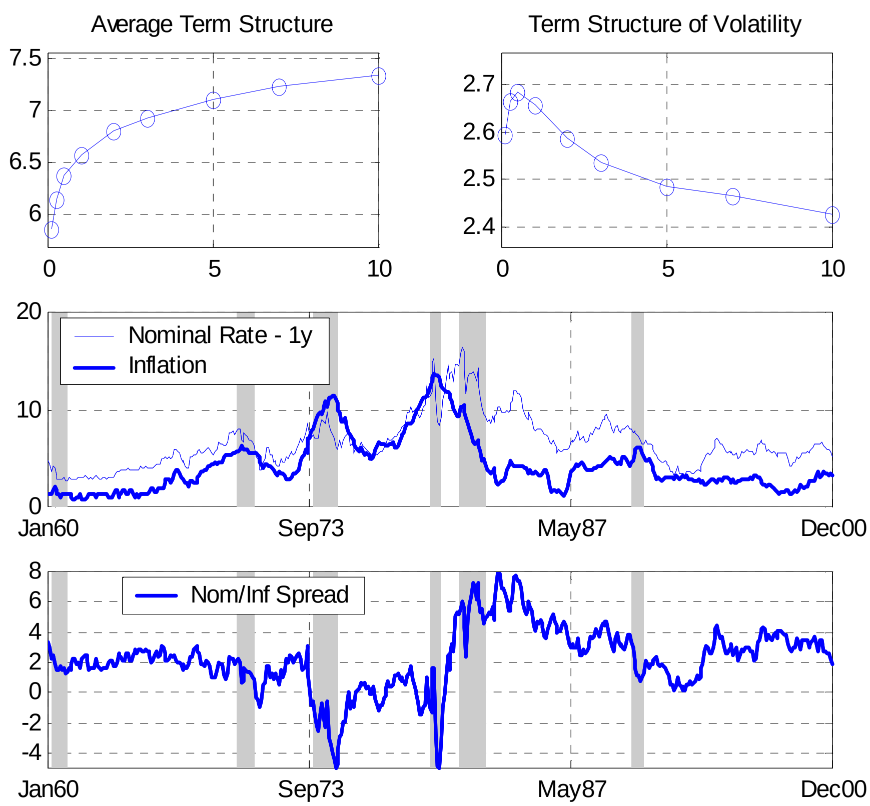

Figure 3: **Summary Statistics**

The figure summarizes the dataset. It is based on 444 monthly observations from January, 1960 to December 2000. In Panel A, pictures 1 and 2 depict the sample mean of the level and volatility of the term structure of nominal interest rates. Panel B shows the evolution of the one year nominal interest rate and CPI Inflation rate. Panel C shows the evolution of the spread between the one year nominal rate and inflation.

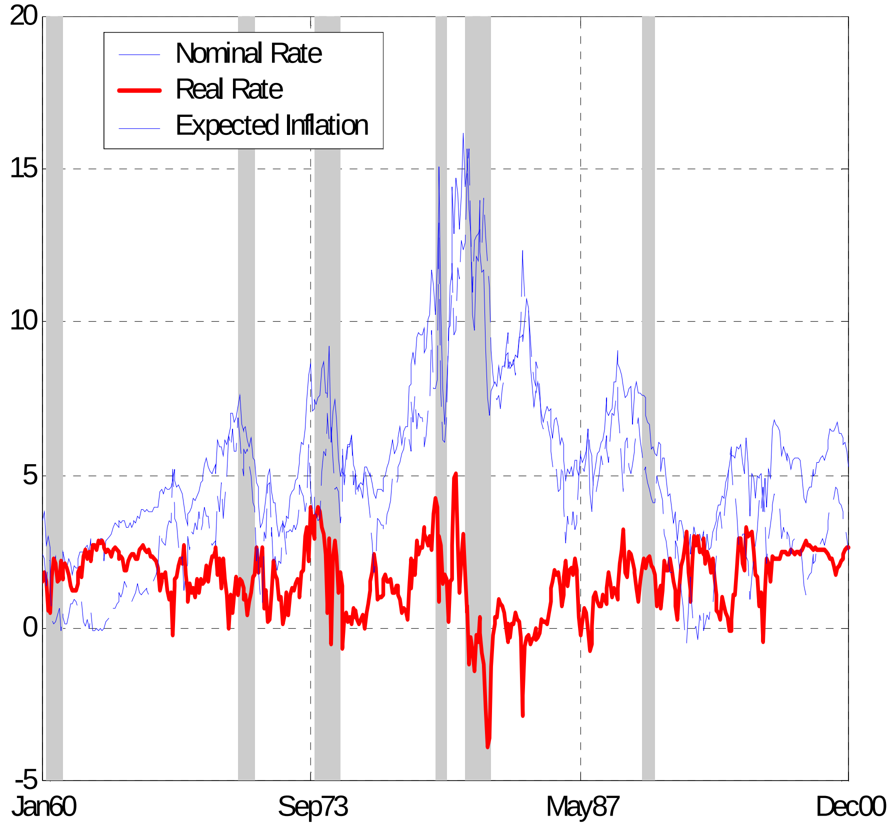

Figure 4: **Real and Nominal Interest Rates and Expected Inflation**
This figure presents the dynamics of the estimated one year nominal rates, the general equilibrium expected level of the inflation rate and the general equilibrium one year real rate. US recessions dates are presented as gray boxes.

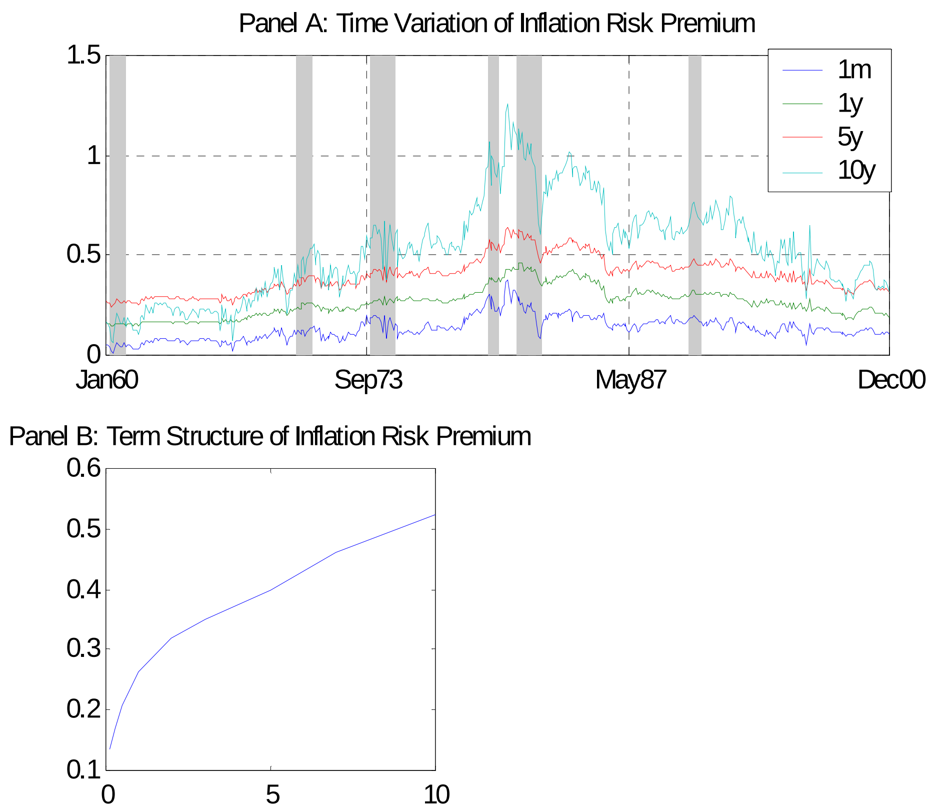

Figure 5: **Inflation Risk Premium**
Panel A shows the time variation of the inflation risk premium between January 1960 and December 2000 US recession periods are marked as gray boxes. Panel B shows the average term structure of inflation risk premium over the entire sample.

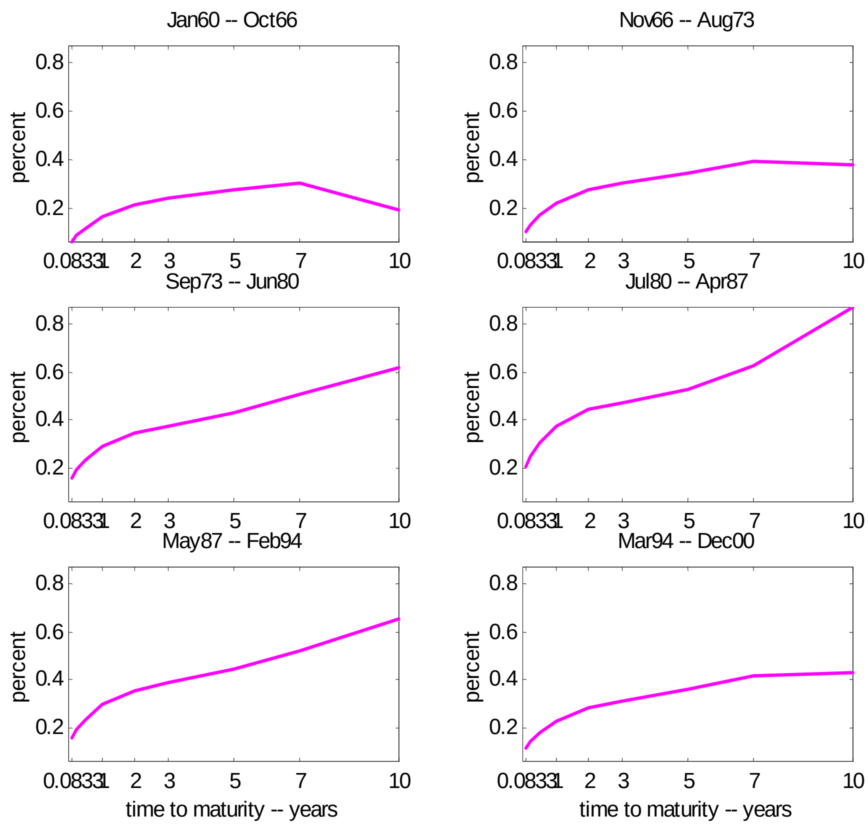

Figure 6: **Term Structure of Inflation Risk Premium by Subperiods**
This figure presents the variation of the average term structure of the inflation risk premium in six subperiods with equal length.

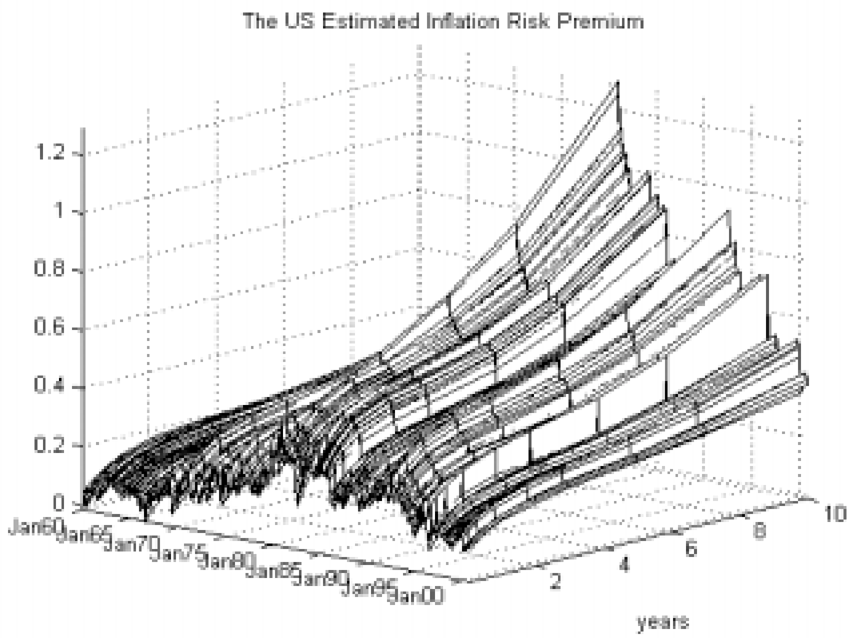

Figure 7: **The US Historical Estimated Inflation Risk Premium**
This figure presents the time-maturity variation of inflation risk premium. Calculation of all series is based on estimated model. Series of each maturity has 492 observation which correspond to monthly time points from January 1960 to December 2000. There are 9 maturity points: 1, 3, and 6 months and 1, 2, 3, 5, 7 and 10 years.